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Keskusteluaiheita - Discussion papers

No. 390

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DURATION, IMMUNIZATION AND MODELS OF THE TERM STRUCTURE OF INTEREST RATES*

* I would like to thank Kari Alho, Vesa Kanniainen, Mikko Niskanen and John Rogers for helpful comments and suggestions. Financial support from the Yrjö Jahnsson Foundation is gratefully acknowledged. I am naturally responsible for any remaining errors or omissions.

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ISSN 0781-6847 27.12.1991

MURTO, Risto, **DURATION, IMMUNIZATION AND MODELS OF THE TERM STRUCTURE OF INTEREST RATES**. Helsinki : ETLA, Elinkeinoelämän Tutkimuslaitos, The Research Institute of the Finnish Economy, 1991. 31 p. (Keskusteluaiheita, Discussion Papers, ISSN 0781-6847; no. 390).

ABSTRACT: The paper presents a survey of modern term structure models and illustrate how they can be used in measuring and hedging interest rate risk. Immunization strategies can differ depending on whether one uses traditional duration measures or modern continuous-time term structure models. We illustrate this using the Vasicek one-factor term structure model and the our parameter estimated using the Finnish money market data. With our parameter estimates, the Vasicek model implies immunization strategies which are clearly different from the hedging implied by the traditional duration model.

KEY WORDS: duration, immunization, interest rate risk, term structure models

1. INTRODUCTION

The main aim of this paper is to present a survey of modern term structure models and illustrate how they can be used in measuring and hedging interest rate risk. There is a considerably body of literature on interest rate hedging concentrating on traditional, deterministic duration measures, see for example Bierwag (1987) and references therein. At the same time there is a distinct literature on the continuous-time term structure models. These models are pricing models, i.e. they give a price for interest-rate sensitive instruments. In many cases the most prominent area of practical applications is interest rate hedging. However, very little is written on these applications. This gap we try to fill.

Furthermore we demonstrate that the implications of the stochastic term structure models can differ essentially from the implications given by the traditional duration models. Especially we analyse the immunizated portfolios implied by the Vasicek one-factor term structure model using parameter estimates estimated in the Finnish money market data.

Increased interest rate volatility and increased sophistication of financial engineering have stimulated the use of many new kinds of interest-rate derivative instruments. The increased use of interest rate contingent claims means financial institutions need more sophisticated approaches to hedge their own balance sheet.

Financial institutions have been using deterministic duration measures to hedge their interest rate risk since the 70's. These traditional duration measures do not, however, offer sufficient guidelines to hedge the interest rate risk associated with derivate instruments, which can be dependent on interest rates in complex ways as is the case, for example, with bond options.

In case of complex (and non-complex) instruments one can use stochastic duration measures, which are built upon sophisticated interest rate models in contrast to the traditional duration measures, which utilize very simple assumptions on the interest rate process. In stochastic duration measures one utilizes the modern literature on the determination of the term structure, see e.g. Brennan and Schwartz (1979), Cox, Ingersoll and Ross (1985) and Longstaff and Schwartz (1991). The basic idea of building an immunizated portfolio is still the same as in the case of traditional duration measures. An immunized portfolio or balance sheet is created by computing the sensitivity of assets and liabilities to changes in interest rates by using a stochastic duration and by structuring the portfolio so that the interest rate sensitivities are equated. In a immunizated portfolio the interest rate risk is eliminated.

The outline is as follows. The next section presents continuous time term structure models. In section 2.1 we discuss models in which the term structure of interest rates are determined. In section 2.2 we present a model where the observed term structure is taken as given. Section 3 illustrates empirically how term structure models can be used to hedge interest rate risk. The last section presents some conclusions and suggestions for practical applications.

Some of the continuous time term structure models are highly mathematical, but we try to emphasize the economic logic of the models. To get more rigorous presentations of the underlying mathematics we refer the reader to Merton (1982) or Duffie (1988).

2. PRICING INTEREST RATE CONTINGENT CLAIMS

2.1 Using Endogenous Term Structure

2.1.1 Partial Equilibrium Models

This section will present a brief exploration of continuous time term structure models. In this Section we will present two one-factor models, namely Vasicek (1977) and Cox, Ingersoll and Ross (1985), which are based on the "traditional" approach to pricing interest rate derivative instruments. In Section 2.2 we present the Heath, Jarrow and Morton (1991) term structure model, which utilizes the observed term structure of interest rates.

The traditional approach begins with a continuous trading economy driven by a finite number of exogenously specified, stochastic state variables. In partial equilibrium (arbitrage approach) models as Vasicek (1977) we make basically ad hoc assumptions on the functional form of market price of risk. In equilibrium approach functional forms of the market price of risk is obtained as part of the equilibrium. We will focus our attention first upon the Vasicek one-factor partial equilibrium term structure model.

In the Vasicek model the underlying factor is the instantaneous interest rate r, which follows the diffusion process:

(1) $$dr = \alpha(r_0 - r)dt + \sigma dW,$$

where α, σ and r_0 = positive constants and
$W(t)$ = Wiener process.

In Ornstein-Uhlenbeck's process drift $\alpha(r_0 - r)$ keeps pulling the process towards its long-term mean. The constant r_0 can be interpreted as the historical average of the instantaneous interest rate. The constant α describes in turn the speed at which the process converges to this mean. The process has a limitation that the instantaneous spot rate can have negative values with positive probability. This probability is small if the interest rate is well above zero and the mean-reverting tendency is sufficiently large.

We assume that the price at time t of a pure discount bond which matures at time T is determined by the spot rate process over the term of the bond. Thus we can apply Ito's formula to the bond's pricing process P(t,T,r(t)) and express the bond price in terms of α, σ and the partial derivatives of the bond price:

(2) $dP = P\ \mu(t,T,r)\ dt + P\ \rho(t,T,r)\ dW,$

where $\mu(t,T,r) = 1/P(t,T,r)\ [P_t + \alpha(r_0 - r)P_r + 1/2\ \sigma^2\ P_{rr}],$

$\rho(t,T,r) = -\ 1/P(t,T,r)\ \sigma\ P_r$ and

P_t, P_r, P_{rr} are first and second partial derivatives of price with respect t and r respectively.

The next step is to use the so-called local arbitrage condition that the price of risk q(t,T,r) (defined as the expected instantaneous excess return above the riskless rate, divided by the instantaneous standard deviation of return) is independent of a bond's maturity, i.e. q(t,T,r) = q(t,r) for all T.

Furthermore Vasicek (1977) specifically assumes that the market price of risk is a constant. This corresponds to the assumption that the mean of the instantaneous rate of return on a bond is the sum of the current instantaneous interest rate and the term premium in the following way:

(3) $\mu(t,T,r) = r + q\ \rho(t,T,r),$

where $\mu(t,T,r)$ = the mean of the instantaneous rate of return on a bond with maturity date T,
r = the instantaneous interest rate,
q = the market price of risk and
$\rho(t,T,r)$ = the standard deviation of the instantaneous rate of return on a bond.

Substituting for μ and ρ from equation (2) gives the so-called fundamental partial differential equation:

(4) $P_t + \alpha(r^* - r)P_r + 1/2\ \sigma^2\ P_{rr} - rP = 0,$

where $r^* = r_0 + q\sigma/\alpha.$

The bond price is the solution to the partial differential equation using the boundary condition that the price of the bond corresponds to its nominal value at maturity. The basic difference with respect to the Black-Scholes option pricing model, the logic of which the Vasicek model follows, can already be seen from (4). The differential equation (4) includes the market price of risk, q, which is a utility dependent parameter. The pricing formula will not be independent of risk preferences as the Black-Scholes option pricing model is.

In the Vasicek case the resulting term structure equation is of the form:

$$(5) \qquad R(t,T) = R(\infty) + (r(t)-R(\infty)) \frac{1}{\alpha M}(1-e^{-\alpha M}) + \frac{\sigma^2}{4\alpha^3 M}(1-e^{-\alpha M})^2,$$

where $M = T - t$ = the maturity of the bond maturing at T and $R(\infty)$ = the yield on a very long-term bond, as $T \to \infty$, is the following:

$$(6) \qquad R(\infty) = r_0 + \sigma q/\alpha - \tfrac{1}{2}\sigma^2/\alpha^2 = r^* - \tfrac{1}{2}\sigma^2/\alpha^2.$$

In order to determine the term structure of interest rates according to the equations (5) and (6) we need an observation of the instantanous interest rate, $r(t)$, and parameter estimates of α, r^* and σ^2. The market price of risk, q, is not needed to estimate separately.

There is a considerable number of other works where the instantaneous spot rate also serves as a state variable. See for example Dothan (1978) and Brennan and Schwartz (1977) and for empirical comparison of alternative specifications of short-term interest rate processes Chan, Karolyi, Longstaff and Sanders (1991). Single-factor models have a drawback in that they imply that the instantaneous returns on bonds of all maturities are perfectly correlated. This feature does not appear in multi-factor models. Brennan and Schwartz (1979) have presented a model where the term structure is determined by two state variables, namely by the instantaneous spot rate and the console rate. Brennan and Schwartz were not able to present the closed form solutions as in (5) for a partial differential equation.

Above we had to assume that the particular form of the risk premium is correct. As pointed out by Cox, Ingersoll and Ross (1985), the arbitrage approach has a limitation in that it provides no way of guaranteeing that the obtained term structure is supported by any underlying economic equilibrium. The functional form of the market price of risk and so the risk premiums are not determined in the model endogenously. The next section briefly presents a model which removes this limitation.

2.1.2 General Equilibrium Models

The Cox, Ingersoll and Ross (1985), CIR, model differs from the arbitrage models in the respect that it is a general equilibrium model where the risk premiums are determined endogenously.

In the CIR model the term structure is determined within a general equilibrium in which firms face a stochastic investment opportunity set and identical agents, with time-additive logarithmic utility, maximise the expected utility of lifetime consumption. Niskanen (1991) presents a good review on the CIR model and its links to the earlier models. CIR describe a number of models, but the empirical analysis has focused on the special case in which there is one state variable and where both this and the instantaneous short rate follow a 'square-root' process:

(7) $$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}\, dW.$$

In (7) κ is the mean reversion coefficient (α in equation 1), θ is the mean of the process (r_0 in equation 1) and σ is a constant governing the scale of changes in r. The main difference compared to equation (1) is that the volatility term incorporates the level of the interest rate. This means that the absolute variance of the interest rate increases when the interest rate itself increases implying a heteroscedastic process. The second difference compared to the process presented by equation (1) is that negative interest rates are now precluded.

In the CIR model the fundamental partial differential equation is as follows:

$$(8) \qquad P_t + \kappa(\theta^* - r)P_r + 1/2\ \sigma^2\ P_{rr} - rP = 0,$$

where $\theta^* = \theta - \lambda r/\kappa$.

The factor λ is the covariance of changes in the interest rate with percentage changes in optimally invested wealth.

The resulting term structure equation is:

$$(9) \qquad R(t,T) = 1/(T-t) * (-\log A(t,T) + B(t,T)r(t)),$$

where

$$(10) \qquad A(t,\ T) = \left| \frac{2\ \gamma\ (e^{[\gamma(T-1)(T-t)]/2} - 1)}{(\gamma + \kappa + \lambda)(e^{\gamma(T-1)} - 1) + 2\gamma} \right|^{2\kappa\theta/\sigma^2}$$

$$(11) \qquad B(t,\ T) = \frac{2(e^{\gamma(T-1)} - 1)}{(\gamma + \kappa + \lambda)(e^{\gamma(T-1)} - 1) + 2\gamma},$$

$$(12) \qquad \gamma = ((\kappa + \lambda)^2 + 2\sigma^2)^{1/2}.$$

As in the Vasicek model, the term structure is determined by the parameters of the instantaneous spot rate process - κ, θ, σ - and the utility dependent parameter λ. One must notice that the volatility parameter, σ, is not the same as in the Vasicek model.

The CIR term structure model has been modificated by several authors. Ahn and Thompson (1988) include the possibility of jumps in the instantaneous spot rate process by adding a Poisson process term in equation (7). Longstaff (1989) on the other hand allows technological change to affect production returns nonlinearly, which implies that the instantaneous risk-free rate follows the following process:[1]

$$(13) \qquad dr = \kappa(\theta - \sqrt{r})dt + \sigma\sqrt{r}\, dW.$$

The main difference is that the restoring force is proportional to the term $(\theta - \sqrt{r})$ rather than $(\theta - r)$ as in previous models.

Perhaps the most interesting extension has been done by Longstaff and Schwartz (LS) (1991). LS developed a two-factor model of the term structure by using the framework of CIR. As in the CIR, Longstaff and Schwartz begin with unspecified factors that affect (the productivity and returns on physical investments), but make transformations to the factors that are easily observable.[2] The first factor is the same as in the previous models, i.e. the short-term interest rate. The second factor is the volatility of the short-term interest rate changes. LS were able to derive closed-form expressions for discount bond prices.

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[1] In the CIR (1985) model the change in production oportunities over time is described by a single state variable. The means and variances of the rates of return on the production process are proportional to this state variable. Longstaff (1989) allows them to be proportional to the nonlinear term X^2, where X is a state variable referred above.

[2] In principle one can construct models that have some macroeconomic variables such as terms of trade or consumption as a state variables. Market participants must, however, use pricing models daily or even more frequently. Compared to the short-term volatility of asset prices statistical observations of macroeconomic variables change much more slowly. We would only have marginal gain by adding new slowly changing state variable to our pricing equation.

Above we presented term structure equations implied by the Vasicek and CIR models. These frameworks can also be used to derive valuation formulas for interest rate dependent derivative instruments.

Cox, Ingersoll and Ross presented an analytical solution for European bond options. Longstaff and Schwartz were also able to present closed form solution for European bond option with two factors. Jamashidian (1989) has solved the price of the European bond option analytically in the case of the Vasicek model. The pricing formula resembles the famous Black and Scholes stock option pricing model with changes in variables. We will postpone the option pricing model presentation until the next section in connection with the Heath, Jarrow and Morton model.

The solution method is usually the same as with the term structure equation. The fundamental p.d.e is solved with the relevant boundary conditions. In the case of the bond call option the relevant boundary condition is the payoff function for the call option at maturity. That is max[0, P - K), where P is the price of the bond and K the strike price of a European call option on a bond.

In many cases there is no analytical solution to the fundamental partial differential equation. In these cases one must use numerical techniques to solve it.

2.2 Using Exogenous Term Structure

In this section we will present the model of Heath, Jarrow and Morton (1991). In some recent papers such as Heath, Jarrow and Morton (HJM) and Turnbull and Milne (1991) the initial term structure of interest rates is taken as given. This differs from previous models, where bond and option prices were determined within the model assuming that the instantaneous spot rate can serve as a sufficient state variable.

The new approach is essentially aimed at pricing interest rate dependent derivative instruments such as options on bonds or options on interest rate futures. However, as Brenner (1989) has shown in the context of HJM-model, we can also derive duration measures for bonds.

A few words on the mathematical approaches to solve the pricing problem are probably worthwhile. In the previous section the pricing formulas were acquired by deriving the so-called fundamental partial differential equation and solving it with a relevant boundary condition. It produces in some cases closed-form solutions for the term structure equation and option pricing formulas, but in many cases pricing formulas must be solved numerically. HJM, as well as Turnbull and Milne (1991) and Stapleton and Subrahmanyama (1991), use the so-called martingale approach to solve the pricing problem.

The martingale (or risk-neutral) representation was first employed by Cox and Ross (1976) for option pricing and was later developed more formally by Harrison and Kreps (1979). According to Harrison and Kreps the asset prices are arbitrage-free only when the prices are martingales under the specific probability measure. The martingale representation yields a price equal to the expected value under the martingale measure of the product of the terminal value times a discount factor that corresponds to rolling over the shortest maturity default-free bonds.

The stochastic discount factor is a particularly natural choice in pricing interest-rate derivative securities. In HJM the discount factor is the price of a money market account rolling over at r(t), i.e

$$(14) \qquad B(t) = \exp\left(\int_0^t r(y)dy\right).$$

Next we define the relative bond price for a T-maturity bond as $Z(t,T) = P(t,T)/B(t)$. The bond price is expressed in units of money market account.

In Section 2.2 we used the so-called local arbitrage condition that the market price of risk is independent of bond's maturity. Heath, Jarrow and Morton establish the equivalence between this and two other alternative conditions given that there are no arbitrage opportunities in the markets. These alternative conditions are: (i) a existence of unique, equivalent martingale measure such that relative bond prices are martingales for all maturities, (ii) the specific form of the forward rate drift under the martingale measure.

Condition (i) implies that there is a unique, equivalent martingale measure so that $E^*[Z(T,T) I F(t)] = Z(t,T)$, where $E^*(.)$ denotes expectations with respect to the probability measure Q^*. The same kind of statement can be also used in a straightforward way in pricing contingent-claims.

Heath, Jarrow and Morton concentrate on the term structure of forward rates and their evaluation in time. The HJM paper's discrete time forerunner was a binomial model by Ho and Lee (1986). The HJM model can be seen as a generalization of the Ho and Lee model.

We will present only HJM's simplest model, where there is only one shock affecting the forward rate and the volatility parameter is a constant. Heath, Jarrow and Morton start by defining the forward rate process. In this case they assume that the forward interest rate follows the following process:

(15) $$df(t,T) = \alpha(t,T)dt + \sigma dW(t),$$

where $f(t,T)$ = the instantaneous forward rate at time t for date T.

Using the martingale condition they write the resulting bond's pricing formula as:

(16) $$P(t, T) = \exp(-\int_t^T (f(0,y) - f(0,t))dy - (\sigma^2/2)t(T-t)^2 - (T-t)r(t))$$

To get more intuition in the above equation we can use the definition of interest rates and forward rates and express the equation in the following form:

(17) $$R(t,T) = F(0,t,T) - f(0,t) + (\sigma^2/2)t(T-t) + r(t),$$

where $F(0,t,T)$ is the forward rate at time 0 applying to the time interval t to T.

Equation (17) states that the (T-t) period interest rate at time t depends on the slope of the initial forward rate curve, $F(0,t,T) - f(0,t)$, the disturbance term, $(\sigma^2/2)t(T-t)$, which grows as the maturity grows, and instantaneous spot rate time t, $r(t)$.

The equation (16) gives an arbitrage-free bond price in an artificial economy, where the discounted bond prices are martingales with respect to the probability measure. In order to price bonds in a real observable world, we would need probability measures governing term structure movements that incorporate risk preferences. However, in order to price contingent claims, which are independent of the preference structure, the artificial economy is sufficient. That is exactly what Heath, Jarrow and Morton do.

We will illustrate the contingent-claim pricing by an example of a European call option on the bond $P(t,T)$ with an exercise price of K and a maturity date t^*, where $0 \le t \le t^* \le T$.

Let $C(t)$ be the value of this call option at time t. The cash flow to the call option at maturity is:

(18) $$C(t^*) = \max [P(t^*,T) - K, 0].$$

Using the martingale pricing condition for contingent claims we can claim that the value of the call at time t can be written as:

(19) $C(t) = E^*(\max [P(t^*,T) - K, 0]B(t)/B(t^*) \mid F(t))$.

The expression (19) is not itself useful in practical pricing purposes, but it can be simplified to the following option pricing formula:

(20) $C(t) = P(t,T)\ N(h) - KP(t,t^*)N(h-\sigma_M(M)^{1/2})$

where $h = [\log(P(t,T)/KP(t,t^*)) +$
$(1/2)\sigma_M^2(t^*-t)]\ \sigma_M(t^*-t)^{1/2}$,
$M = T-t^*$,
σ_M = variance of the instantanous return on the forward price of a T-maturity bond and
N(.) = the cumulative normal distribution.

The resemblance between equation (20) and the Black-Scholes formula is clear. The value of the bond option is equal to the Black-Scholes formula, where the bond price, P(t,T), replaces the stock price, P(t,t*), plays the role of exp(-r(T-t)) in the B-S model and σ_M replaces the volatility of the stock.

This particular bond option pricing formula has also been derived by several other authors. Turnbull and Milne used a general equilibrium model. They present closed form solutions e.g. for European options written on Treasury bills, interest-rate forward contracts, interest-rate futures contracts, Treasury bonds and interest-rate caps. Jamshidian (1990) derived the bond option formula in the Vasicek term structure framework. Stapleton and Subrahmanyam (1991) have also derived a bond option pricing formula.

A final note on the division of endogenous and exogenous term structure models used above is justified. In practice the difference is not necessarily so fundamental. Also Vasicek and

CIR models can be expressed in terms of the observed term structure of interest rates and the observed term structure of spot and forward interest rate volatilities; see Hull and White (1990).

3. HEDGING INTEREST RATE RISK

3.1 Deterministic and Stochastic Duration Measures

In the previous section we made a brief exploration of the continuous time term structure models. These models can play a prominent role in interest rate hedging. They offer the possibility of using consistent models in pricing of different interest rate sensitive assets and derivative instruments and calculating duration measures. That helps to assess and hedge interest rate risk. We will show that hedged portfolios implied by the term structure models can differ essentially from the portfolios implied by the traditional duration measures.

We will start with a discussion of the traditional duration measures. The modified duration will serve as an example of many deterministic duration measures. Duration measures are aimed to measure interest rate risk, i.e. risk arising unexpected term structure movements. We will see that duration, as used in literature, is essentially an elasticity measure of bond price sensitivity with respect to the change in some index of the term structure. We illustrate that the elasticity measure, $-P_r/P$, is a valid risk measure also in continuous-time term structure context.

Most of the duration literature has concentrated on deterministic duration measures, see for example Bierwag (1987). Macaulay (1938) presented duration measure as a weighted average of a coupon stream and principal payment. The role of the duration as a proxy for interest rate risk was originally proposed by Hicks (1939). The modified duration derived by Hicks and afterwards by Hopewell and Kaufman (1973) is attainded by computing the differential of the bond price with respect to the yield to maturity. The bond price is equal to the sum of the present values of the stream of coupon payments and of the final payment at maturity:

$$(21) \qquad P(t,T) = \sum_{t=1}^{T} \frac{C(t)}{(1+y)^t} + \frac{A}{(1+y)^T},$$

where $C(t)$ = coupon payment at time t,
A = final payment at maturity,
y = yield to maturity.

Differentiating (2) with respect to y we obtain the following equation:

$$(22) \qquad dP = - \left[\sum_{t=1}^{T} \frac{tC(t)}{(1+y)^t} + \frac{TA}{(1+y)^T} \right] dy/(1+y).$$

This can be written as

$$(23a) \qquad dP/P = - D\, dy/(1+y) = - D_{mod}\, dy \quad \text{or}$$

$$(23b) \qquad D_{mod} = - dP/dy\ (1/P) \quad \text{and}$$

$$(23c) \qquad D = \frac{\left[\sum_{t=1}^{T} \frac{tC(t)}{(1+y)^t} + \frac{TA}{(1+y)^T} \right]}{\left[\sum_{t=1}^{T} \frac{C(t)}{(1+y)^t} + \frac{A}{(1+y)^T} \right]}$$

where D is the duration and $D_{mod} = D/(1+y)$ the modified duration.

It is well known that the modified duration formula holds only when the term structure is flat and the stochastic process restricts interest rate movements to parallel changes.
There are several modifications that allow different discrete

interest rate changes, see Khang (1979) for a particular example. These duration measures, however, still have the drawback that they either rely on very resrictive assumptions or even worse allow arbitrage opportunities as pointed out by Ingersoll, Skelton and Weil (1978).

Next we investigate the proper duration measure in continuous-time term structure framework, where the arbitrage opportunities are precluded. We verify that the semi-elasticity, i.e. P_r/P, is a proper measure for risk also in this context. The elasticity is now calculated based on the stochastic term structure model.

We use an example of the one-factor model starting from the following general diffusion process:

(24) $$dr = f(t,r)\,dt + \sigma(t,r)\,dW,$$

where $f(t,r)$ is drift and $\sigma(t,r)$ variance. We have the Vasicek model if we define $f(r,t) = \alpha(r_0 - r)$ and $\sigma(t,r) = \sigma$. Respectively we have the CIR one-factor model if we define $\sigma(t,t) = \sigma\sqrt{r}$.

Using Ito's lemma we can state that bond price satisfies the following equation:

(25) $$dP/P = \mu(t,T,r)\,dt - \rho(t,T,r)\,dW,$$

where $\mu(t,T,r) = P_r/P + f(t,r)P_r/P + 1/2\ \sigma^2\ P_{rr}\ /P$ and

$$\rho(t,T,r) = -\ \sigma(t,r)\ P_r/P.$$

Since $\sigma(t,r)$ is common to all bonds, $-\ P_r/P$ is a valid measure of risk associated with the unexpected change in instantaneous interest rate.[3]

Next we extent our analysis to cover both assets and liabilities. If we define the price of the liabilities to be P^L and the price of the assets P^A, the stochastic component of the return of the portfolio will be

(26) $(\sigma \, P^A{}_r/P^A - \sigma \, P^L{}_r/P^L) \, dW.$

By selecting assets and liabilities so that

(27) $P^A{}_r/P^A = P^L{}_r/P^L,$

total return will be independent of stochastic component dW and total return must be zero. We state that the position is immunized, when assets and liabilities are structured so that the stochastic component is eliminated from our portfolio. Portfolio is immunizated when equation (27) holds. In a immunizated portfolio the interest rate risk is eliminated.

In practice immunized portfolios are created by computing the sensitivity of assets and liabilities with respect to changes in factors using duration measures, and structuring the portfolio so that the interest rate sensitivities are equated. In many cases we also require that the value of the liabilities equals

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[3] Brenner (1989) has shown that in the context of the HJM model the local percentage change in bond price can be presented as:

(25b) $dP/P = \mu(t,T) \, dt + (-\int_t^T \sigma(t,v) \, dv) \, dW,$

where $\sigma(t,v)$ is the forward rate volatility coefficient. The drift, $\mu(t,T)$, depends on the initial forward rate curve and the forward rate process' drift and volatility coefficient. The local percentage change due to the unexpected change in forward rate curve is the integral of the forward rate volatility coefficient over the remaining life of the bond. As discussed in Section 2.2 HJM price bonds only in the artificial economy. However, as the forward rate volatility coefficient is independent of preference restrictions, we can acquire duration measures also in HJM-model.

the value of the asset, $P^A = P^L$. Risk is then eliminated, when $P^A{}_r = P^L{}_r$.

Above we used a one-factor model to investigate a valid risk measure. The example can be generalized to a n-factor case straightforwardly, see for example Peltokangas (1991, pp. 40 - 41) on a two-factor case. Now we are ready to define a general stochastic duration measure.

The stochastic duration is a bond price semi-elasticity with respect to a change in the underlying factor f_i:[4]

$$(28) \qquad D_i(r,t,T) = - \frac{\partial P(r,t,T) \; / \; \partial f_i}{P(r,t,T)}$$

where $P(r,t)$ = price of the bond,
f_i = factor i.

We use the term structure models to calculate derivatives and prices of different kind of instruments.

Recall that in equations (24) and (25) we did not use any particular asset specific restrictions such as boundary conditions. In fact we can derive stochastic duration measures for any asset X_i and factor f_i given a specific stochastic model for this asset price. For example, if we substitute the price of the bond, P, by the price of the European call option, C, we can derive a duration measure for this particular option.

In the one-factor model case, where the factor is usually the instantaneous interest rate as above, the duration gives a bond's (or other asset's) semi-elasticity with respect to the

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[4] This is not the only possibly definition of duration. For example, Cox, Ingersoll and Ross (1979) and Brenner (1989) call the price elasticity of a security as a basis risk. A security's (coupon bond) duration is defined in turn as the maturity of the pure discount bond with the same basis risk. See more from CIR and Brenner.

marginal change in the instantaneous interest rate. In the multi-factor model we have more than one duration measure. For example in the Longstaff-Schwartz (1991) two-factor term structure model, where the factors are the instantaneous interest rate and the volatility of the instantaneous interest rate changes, we can derive exposures to changes in both the interest rate and the volatility. The portfolio is immunized when both exposure measures are separately equated on both sides of the balance sheet.

The sensitivity of a portfolio to the factor change is simply the sum of each asset's sensitivities to that factor weighted by the asset's share in the value of the portfolio. The portfolio is immunized when the weighted sum of elasticities is equal zero.

Immunization can be seen as analogous to the Black-Scholes continuous hedge strategy in option pricing, where portfolio proportions depend on the sensitivity of the option price to a change in the underlying stock (S). In the continuous trading economy the perfect hedge is acquired by restructuring the portfolio continuously.

The duration measure in turn has a close counterpart in the option hedge literature, namely delta $\partial P/\partial S$, which measures option price sensitivity to the underlying asset's changes. See more in Hull (1989) chapter 8. In the option literature there are several other partial derivatives with respect to other model variables. One of the most important is gamma, defined as $\partial^2 P/\partial S^2$, which is an option price's second derivative with respect to the stock price. It measures the delta's elasticity with respect to the stock price movements. In the interest rate hedging literature the corresponding measure is convexity. Convexity measures the rate at which the duration itself will change as the factor changes.

The mathematical definition of convexity is:

$$(29) \qquad C_i(r,t,T) = - \frac{\partial^2 P(r,t,T) \ / \ \partial f_i^2}{P(r,t,T)} ,$$

where the variables are the same as above.

If convexity is small, duration changes only slowly. If convexity is large, however, duration is sensitive to the changes in the underlying factor values. In this case the adjustments to keep portfolio hedged need to be made frequently.

The fact that in practice trading is done discretely and the possibility that a factor's values can sometimes experience discrete jumps require the measurement of exposure of a measure of exposure. Bond elasticity, being a marginal rate of change, is not exactly equal to the total percentage price change for a sudden jump in a factor even it can approximate it. The convexity can measure how risky our position is with respect to the sudden jumps.

Above we already pointed out that hedging is essentially dynamic. We need to restructure the portfolio as the factor values change. Moreover the duration will change as the maturity of fixed income assets changes. The position, which was hedged last month will not necessarily be hedged today any more as the maturity of asset has changed. Hedging requires active management. As Maloney and Logue (1989) have demonstrated active management can produce significant transactions costs, which can affect the return on the immunized portfolio.

3.2 Stochastic Duration Measures in Practice

We will illustrate the use of stochastic duration measures by simple examples. In the first case, we suppose that our aim is to immunize 7-year discount bond debt with 5- and 10-year discount bonds. We require that in our portfolio the value of assets equals the value of the liabilities at time t (i.e when we hedge). Our task is find how much of these bond we must buy.

Let N_5^1 and N_{10}^2 be the amounts of 5 and 10-year bonds respectively. The amounts N_5^1 and N_{10}^2 for a given instantaneous spot rate level and term structure are obtained by solving:

(30a) $N_5^1 P(5) + N_{10}^2 P(10) = 100P(7)$

(30b) $N_5^1 P_r(5) + N_{10}^2 P_r(10) = 100P_r(7)$

To derive the hedged portfolio we use both stochastic and deterministic duration measures. The stochastic duration will be based on the Vasicek term structure model. For the deterministic model $P(r_T, T) = \exp(-r_T T)$ and hence $P_r(r_T, T) = -T \exp(-r_T T)$. We suppose that the observed term structure corresponds exactly to the term structure implied by the Vasicek term structure model. We also assume that under the deterministic model the changes of different maturity interest rates change by an equal amount. Thus we allow horizontal term structure movements.[5]

In order to apply the Vasicek model we need estimates for r^*, σ^2 and α. We use parameters estimated from weekly Helibor-rates. The estimated parameter values are $r^* = 0.1236$, $\sigma^2 = 0.0293$ and $\alpha = 0.5467$. See equation (5) and (6) in Section 2.1. The estimation as also the data is explained in more detail in the appendix. Furthermore we assume that the level of the instantaneous interest rate is 0.10.

The results from the immunization simulations are presented in table 1. The hedged portfolios corresponding to both the Vasicek model and the simple deterministic duration are presented. The first two rows present the numbers of the 5- and 10-year discount bonds that will immunize a 7-year target bond. The

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[5] An alternative way would be to assume flat term structure, when under the deterministic model the term structure can be solely determined by the maturity and the instantanous interest rate. The change in the instantanous interest rate would induce horizontal changes in the term structure. In this setting the duration, presented by the equation (23b), is a valid risk measure.

stochastic model gives a relatively greater investment in the long-term asset.

In Figure 1 we have N_T^1 and N_{10}^2 as a function of T, the maturity of the first bond. N_{10}^2 is zero when T=7 i.e. when the first bond's maturity equals the target bond's maturity. When T is less than seven years, N_{10}^2 is always greater than N_T^1. Both N_T^1 and N_{10}^2 will start to increase rapidly in absolute terms as N_T^1 approaches the maturity of the second bond.

Table 1. Amounts of two discount bonds to immunize a target bond. The face value of a target bond is assumed to be 100. The parameter values used in the Vasicek model are $r^* = 0.1236$, $\sigma^2 = 0.0293$ and $\alpha = 0.5467$.

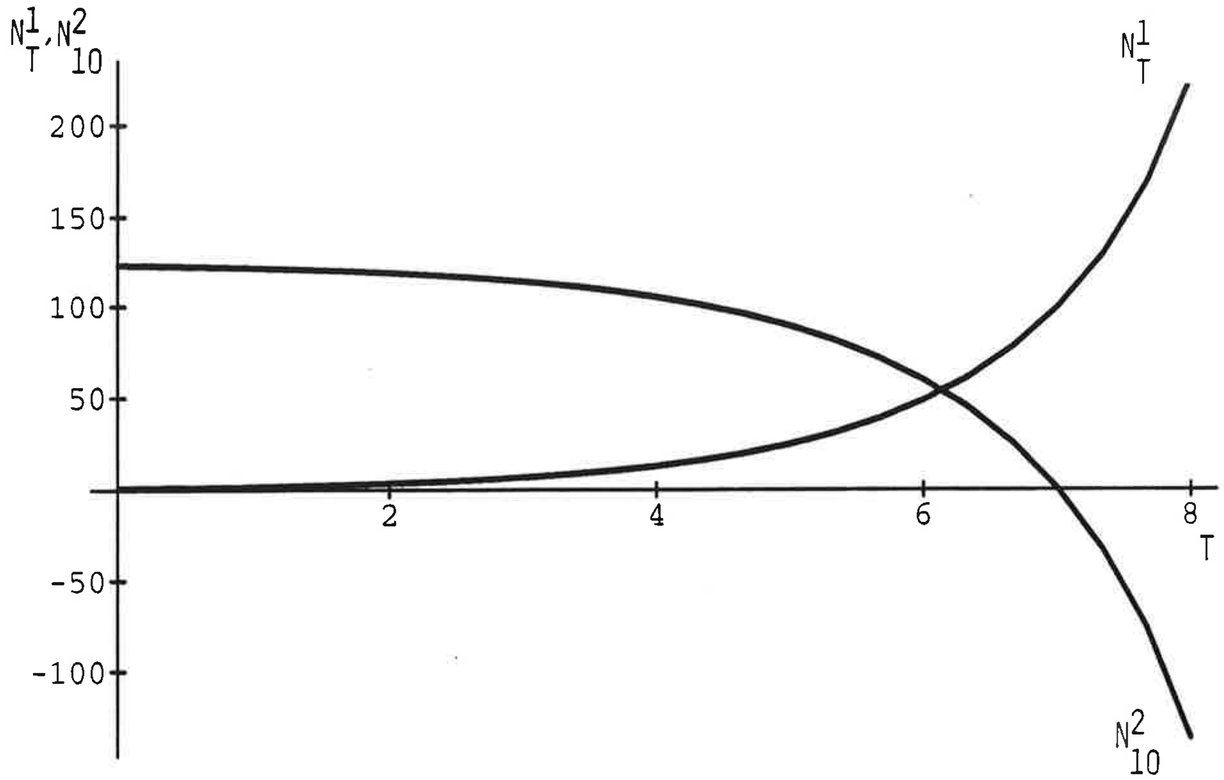
The model	Amount of bond 1	Amount of bond 2	Target bond
	N_5^1	N_{10}^2	N_7^3
Vasicek	24.74	89.16	100
Simple	51.39	50.15	100
	$N_{1/2}^1$	N_2^2	N_1^3
Vasicek	54.40	47.13	100
Simple	64.35	35.59	100
	$N_{1/2}^1$	N_1^2	N_2^3
Vasicek	-115.43	212.17	100
Simple	-180.83	281.00	100

In the second case we are interested in hedging a 1-year bond with a combination of 1/2 and 2-year bonds. The difference between the stochastic and deterministic models is the same as above: the Vasicek model gives a relatively greater investment in the longer-term asset.

In the last example we hedge a 2-year target bond with

1/2- and 1-year bonds. In this case the immunized portfolio is acquired by going short with 1/2-year discount bonds and going long with 1-year bonds.

Figure 1. Amount of bond number one as a function of T



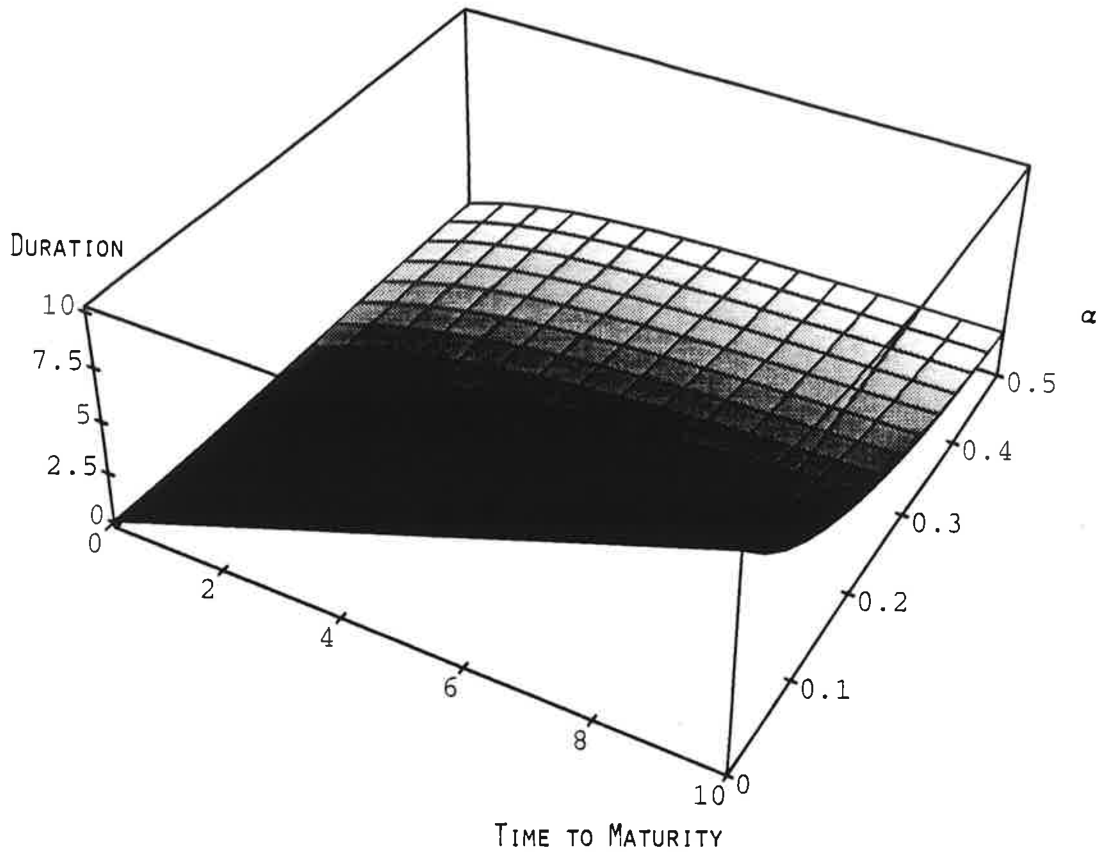
Under the deterministic model a small change in the instantaneous interest rates induces a movement in the level of the entire term structure. This implies that the price of the long-term bond is very sensitive to interest rate changes. Under the stochastic model the sensitivity of the long-term bond price is smaller, which can be seen from durations. Under the deterministic model the duration of the discount bond is simply the time to maturity. In the Vasicek model the duration of 5-, 7- and 10-year bonds with above described parameter values are

1.71, 1.79 and 1.82 respectively. The riskiness measured by the duration is increasing in maturity much more slowly in the stochastic model than in the deterministic model. The reason is that in the Vasicek model the instantanous interest rate keep pulling back to it's long-term mean when α is positive and different from zero. The unexpected change in the instantanous interest rate will not affect the long-term interest rates any greater extent, because the instantanous interest rate is expected to return toward it's long-term mean.

The figure 2 shows the relationship between the value of α and and duration for different maturities. As the value of the α approaches zero the difference between the durations implied by the stochastic and deterministic models dimish. For example, when α is 0.001 the duration of the 5-year discount bond in the Vasicek model is 4.99. In this case the stochastic process of the instantanous interest rate is close to the random walk. The unexpected changes in the short-term interest rates are not expected to perish so that the term structure moves very much in the same manner as under the deterministic model.

Now we can discuss the relative advantages of using stochastic duration measures as compared to the deterministic duration measures. One major advantage is that one particular term structure framework gives us consistent models to price and hedge different interest-rate sensitive instruments. When hedging an investor can use the same parameter estimates and the same set of factors in calculating elasticities for different instruments. This ensures that assumptions, for example, on interest rate volatility are consistent when pricing and hedging different instruments.

Figure 2. The relationship between the value of α, maturity and duration.



The stochastic duration measures are also natural when hedging complex instruments such as options on bonds or options on interest rate futures. Furthermore traditional duration measures do not provide any guidance in hedging against possible second factors as the risk of volatility changes in Longstaff and Schwartz-model. We can also simulate the impact of parameter changes in our portfolio. For example, in the Vasicek model we can study how our hedging will change as the estimate of α, σ^2 or r^* changes.

In the above examples we assumed that the observed term structure corresponds exactly to the term structure implied by the Vasicek model. This is hardly a general case. The theoretical and observed term structures will differ in most cases, even though we are fitting the model to the observed data. This can cause problems in immunization. Even thought the equations (30a) and (30b) hold for our portfolio, the market value of assets and liabilities can differ because theoretical and observed bond prices differ. It can be that equation (30a) no longer holds, when theoretical prices are replaced with observed market prices. One possible solution is to use market prices in equation (30a) and stochastic model in equation (30b). Now the drawback is that restrictions implied by these equations can be inconsistent for the given parameter values.

The basic idea of immunization appears to be simple, but applications of stochastic duration measures can be quite complex. As mentioned in Section 2.1, in many cases there is no analytical solution to the fundamental partial differential equation, in which case one must use numerical methods. The practioners often prefer the simple models that are cheaper and less time consuming, especially if they have a plethora of different instruments to price and hedge.

The further complication as compared to the deterministic duration measures is that the precision of stochastic duration measures relies partly on the parameter estimates. The accuracy of immunization is dependent on our estimation procedure and the stability of the parameters.

Needless to say the stochastic duration measure essentially depends on the specification of factors and factor risk premiums. How many factors we need, what these factors are and how they evolve over time are basically empirical questions.

Traditional duration measures have performed quite well when compared to the stochastic duration measures in empirical comparisons. For example Brennan and Schwartz (1983) compared stochastic duration based on the two-factor term structure model to the traditional duration model. In immunization simulations the traditional and stochastic duration models yielded quite similar results. This defends the use of traditional duration measures in basic hedging situations where the portfolio does not include complex instruments.

The literature concerning the relative performance of different stochastic measures is, however, still quite limited. For example, it is an open research question how much better immunization can be acquired when the volatility risk is also incorporated in hedging as LS-model two factor model imply.

4. CONCLUSIONS

We have presented several basic approaches to modeling prices of interest rate sensitive instruments. We noticed that there are basically two ways of pricing these assets. We can either determine the term structure of interest rates within our model or take the term structure of interest rates as given. Bond prices can only be determined in the first approach, but both ways lead to pricing formulas for derivative instruments. The choice of a proper framework is in some cases almost trivial as both avenues can result in the same kind of pricing formulas (e.g. Black-Scholes formula).

The term structure models can be used to assess and hedge interest rate risk. We referred to duration measures based on the stochastic term structure models as stochastic duration measures. The stochastic duration measures have clear theoretical benefits over the traditional duration measures. The stochastic duration models preclude arbitrage opportunities. We can use consistent models in assessing interest rate risk using the stochastic duration measures. Furthermore, in the case of two factor models as that of Longstaff and Schwartz we can get price sensitivies also for other factors than short-term spot rate. We can assess interest rate risks arising from different sources.

Immunization strategies can differ essentially depending on whether one uses traditional duration measures or modern stochastic term structure models. We used the Vasicek one-factor term structure model and the parameters estimated using the

Finnish money market data. With our parameter estimates, the Vasicek model implies immunization strategies which are clearly different from the hedging implied by the traditional duration model. In some parameter values the difference between the implications from the models can however diminish. This happens when the instantanous interest rates follows a random walk.

The drawback of stochastic duration measures is that the resulting formulas can be considerably more complicated than in traditional duration measures. We must also rely on parameter values in most cases, which must be estimated from the data. Traditional duration measures can also yield satisfactory results in practical situations. On the other hand there are quite many open research questions concerning the stochastic duration measures which can be answered only by further empirical work.

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APPENDIX

The estimation was based on Helibor-rates. The Bank of Finland calculated daily Helibor-rates (Helsinki Intervank Offer Rates) for 1, 2, 3, 6, 9 and 12 month maturities as the average bid rate for the bank's CDs quoted by the five large banks. The data is weekly and starts from 6.1.1987 and ends at 19.02.1991. There are 209 observations on each maturity.

The parameters were estimated by the GMM. The instruments used were lagged one-month interest rates. We estimate the model with data in the level form in order to have all parameters estimated. When using date in the first differences we have only one parameter estimated, namely α.

Table 1 represent the results from estimation of the Vasicek model. Table display parameter estimates, their t-statistics, the GMM minimized criterion (χ^2) value, its associated degrees of freedom and p-value.

Table 1. Estimating the Vasicek Model

Parameter estimates			Goodness-of-fit		
α	σ^2	r*	χ^2	d.o.f	p-value
0.5467 (19.21)	0.0293 (6.45)	0.1236 (32.66)	143.30	22	.000

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