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ABOUT "EXACTNESS" OF INDEX NUMBER FORMULAS IN DEMAND WORLDS

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1. Introduction

Afriat (1972), Diewert (1976) and others have defined and used the concept exactness of an index number in the context of demand or production theory. Their definition goes roughly as follows. Let $u(q)$, $u: \mathbb{R}^n_{++} \to \mathbb{R}$, be a utitity function and $C(p,q^*) = \min \{C | C = p \cdot q \;\&\; u(q) \geq u(q^*)\}$ be the cost function or minimum expenditure function for given reference prices $p \in \mathbb{R}^n_{++}$ and quantities $q^* \in \mathbb{R}^n_+$ which determine the reference utility level. Define the economic (or true) price index $P(p^1,p^0;q^*) = C(p^1,q^*)/C(p^0,q^*)$ as the ratio of minimum expenditures to bye the utility (or welbeing, level of living) determined by q^*. $P(p^1,p^0;q^*) = P(p^1,p^0;\bar{q})$ for all p's when q^* and $\bar{q}$ are indifferent, $q^* \sim \bar{q}$ or $u(q^*) = u(\bar{q})$. The preferences are homothetic if $q^* \sim q \Leftrightarrow kq^* \sim kq$ for all $k \in \mathbb{R}_{++}$. This can be stated also using the demand system $h(p,C)$ defined by: $h(p,C) = q \Leftrightarrow p \cdot q \leq C \;\&\; u(q) \geq u(\bar{q})$ for all $\bar{q} \in \mathbb{R}^n_{++}$ such that $p \cdot \bar{q} \leq C$. Preferences are homothetic if and only if for all $k \in \mathbb{R}_{++}$: $h(p,kC) = kh(p,C)$. For homothetic preferences $P(p^1,p^0; q^*)$ is invariant not only if indifferent $\bar{q}$:s are inserted in the place of q^* but $P(p^1,p^0; q^*) = P(p^1,p^0; \bar{q})$ for all $q^*,\bar{q} \in \mathbb{R}^n_+$

In the homothetic case a function ("index") $f: \mathbb{R}^{4n}_{++} \to \mathbb{R}_{++}$, $\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} \to f\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix}$ is an exact price index for the given preferences if

$$(1) \qquad f\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = P(p^1,p^0; q^*)$$

for all $\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} \in \mathbb{R}^{4n}_{++}$ such that $q^0 = h(p^0, p^0 \cdot q^0)$ and $q^1 = h(p^1, p^1, q^1)$. This means that our f-function gives for all equilibrium situations (p^0, q^0), (p^1, q^1) just the correct value of the true price index. If (1) holds for a function f it is often thought (and used) to rationalize the use of that f as a price index. Index number theorists seem to think that it is a kind of merit for a function f to satisfy (1) for some homothetic preferences. The general idea seems to be that those f's that are exact for some preferences (or particularly for some flexible families of preferences) are in some way more suitable index number formulas to be used in more general situations also than those f's which are not exact for any preferences (or are exact only for very restrictive preferences). We will demonstrate that the fulfillment of (1) for a f by no means quarantees its usefulness in other situations. The function f must satisfy many other properties than (1) to be a useful price index number formula in more general situations, e.g. if the data is generated by some other demand mechanism (preferences) than implicite in (1) or if p's and q's in $f\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix}$ change freely. These other properties that f should satisfy are investigated in descriptive (or axiomatic, atomistic, test-theoretic, statistical) index number theory, see e.g. Fisher (1922), Eichhorn (1976), Eichhorn & Voeller (1976), Allen (1975) or Vartia (1976).

2. Examples, the Cobb-Douglas case

It is well-known that the weighted geometric average $\Pi(p_i^1/p_i^0)^{w_i^0}$ of the price relatives (weighted by the old value shares $w_i^0 = p_i^0 q_i^0 / p^0 \cdot q^0$) is exact for the "Cobb-Douglas case" or when $u(q) = \Pi q_i^{c_i}$, where c_i:s are some non-negative constants. But because here $w_i^0 = w_i^1 = p_i^1 q_i^1 / p^1 \cdot q^1$ (the new value shares), where $q_i^1 = h^i(p^1, p^1 \cdot q^1)$, also $\Pi(p_i^1/p_i^0)^{w_i^1}$ and $\Pi(p_i^1/p_i^0)^{\frac{1}{2}(w_i^1+w_i^0)}$ are exact. Any function f: $\mathbb{R}^{4n}_{++} \to \mathbb{R}_{++}$ coinciding with $\Pi(p_i^1/p_i^0)^{p_i^0 q_i^0 / p^0 \cdot q^0}$ when (p^0, q^0) and $p^1, q^1)$ are equilibrium points is exact here.

Let's investigate more carefully the conditions determining equilibrium points (p,q) in the Cobb-Douglas case. The demand system $h(p,C) = (h^1(p,C),\ldots,h^n(p,C)) = (h^i(p,C))$ may be written in numerous forms which complicates the issue.

Its first representation uses only the parameters of $c = (c_i)$ of the utility function $u(q) = \Pi q_i^{c_i}$:

(2) $$h_i(p,C) = \frac{C}{p_i}\left(\frac{c_i}{\Sigma c_j}\right)$$

This is equivalent to

(3) $$\frac{p_i h_i(p,C)}{C} = \frac{c_i}{\Sigma c_j},$$

which shows the utmost speciality of the CD-case:

by (3) the value share $w_i = p_iq_i/p\cdot q = p_iq_i/C$ is constant $(=c_i/\Sigma c_j)$ for all equilibrium points $(p,q) = (p, h(p,C))$. Thus if (p^0,q^0) is any equilibrium point we know that $w_i^0 = p_i^0q_i^0/p^0\cdot q^0 = c_i/\Sigma c_j$ and we may reparametrize (2) and (3) by

(4) $$h_i(p,C) = \frac{C}{p_i} w_i^0$$

(5) $$p_ih_i(p,C)/C = w_i^0.$$

This is the natural way to think the Cobb-Douglas case: it is the case where the demand system is determined by the constancy of the value share, i.e. by (5). From now on let (p^0,q^0) be a fixed pair in $\mathbb{R}^2_{++}$ and $w^0 = (w_i^0) = (p_i^0\, q_i^0/p^0\cdot q^0)$. Now we can determine in a compact way all the "index number formulas f" which are exact in the Cobb-Douglas case. They are functions $f\colon \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ which satisfy

(6) $$f\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \prod_{i=1}^{n} (p_i^1/p_i^0)^{p_i^0q_i^0/p^0\cdot q^0}$$

when $w_i^1 = p_i^1q_i^1/p^1\cdot q^1 = p_i^0q_i^0/p^0\cdot q^0 = w_i^0$ and are otherwise arbitrary. For instance the following functions satisfy (6):

"The logaritmic Paasche"

(7) $$f_2\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \Pi(p_i^1/p_i^0)^{p_i^1q_i^1/p^1\cdot q^1} = \Pi(p_i^1/p_i^0)^{w_i^1}$$

(8) $$f_3\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \Pi(p_i^1/p_i^0)^{\sqrt{w_i^1w_i^0}}$$

"Vartia-Sato"

$$(9) \qquad f_5\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \Pi(p_i^1/p_i^0)^{L(w_i^1,w_i^0)/\Sigma L(w_j^1,w_j^0)}$$

$$= \frac{p^1 \cdot q^1}{p^0 \cdot q^0} \Big/ f_5\begin{pmatrix} q^1 & p^1 \\ q^0 & p^0 \end{pmatrix}$$

where $L(x,y) = (x-y)/\log(x/y)$ is the logarithmic mean of positive numbers x and y, see Vartia (1976).

"Vartia I"

$$(10) \qquad f_6\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \Pi(p_i^1/p_i^0)^{L(p_i^1 q_i^1,p_i^0 q_i^0)/L(p^1 \cdot q^1,p^0 \cdot q^0)}$$

$$= \frac{p^1 \cdot q^1}{p^0 \cdot q^0} \Big/ f_6\begin{pmatrix} q^1 & p^1 \\ q^0 & p^0 \end{pmatrix}$$

"Törnqvist"

$$(11) \qquad f_7\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \Pi(p_i^1/p_i^0)^{\frac{1}{2}(w_i^1+w_i^0)}$$

$$(12) \qquad f_8\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \Pi(p_i^1/p_i^0)^{w_i^0 (w_i^1/w_i^0)^{10}}$$

$$(13) \qquad f_9\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \Pi(g(w^1,w^0)p_i^1/p_i^0)^{w_i^0},$$

where $g\colon \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ is any function satisfying $g(w^0,w^0) = 1$. Also all the "factor antitheses" of these functions, i.e. functions $f_{10},\dots,f_{17}$

$$(14) \qquad f_{8+i}\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \frac{p^1 \cdot q^1}{p^0 \cdot q^0} \Big/ f_i\begin{pmatrix} q^1 & p^1 \\ q^0 & p^0 \end{pmatrix}, \qquad i = 2,\dots,9$$

satisfy (6). Note that because (9) and (10) satisfy the factor reversal test (for all arguments) we have $f_9 = f_{17}$ and $f_{10} = f_{18}$ for all $\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} \in \mathbb{R}^{4n}_{++}$, not only for $\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} \in CD$, where $CD = \{\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} \in \mathbb{R}^{4n}_{++} \mid p^1_i \cdot q^1_i / p^1 \cdot q^1 = p^0_i q^0_i / p^0 \cdot q^0\}$ is the subset of $\mathbb{R}^{4n}_{++}$ where $w^0_i = w^1_i$ and where all the functions (7)-(14) coincide. If $K: \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ is any function such that $K(x,x) = x$ for all $x \in \mathbb{R}_{++}$ (e.g. $K(x,y)$ is some <u>mean</u> of x and y) then any function

$$(15) \qquad f_{(i,j)}\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = K\left(f_i\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix}, f_j\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix}\right), \; i,j = 1,\dots,14, \; i = j,$$

is a function from $\mathbb{R}^n_{++}$ to $\mathbb{R}_{++}$ and satisfies (6). Take e.g. $K(x,y) = x + 1946 \sin(x,y)$, for which $K(x,x) = x + 1946 \sin(0) = x$, choose $(i,j) = (3,15)$ and we have:

$$(16) \qquad f_{(3,15)}\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \Pi(p^1_i p^0_i)^{\sqrt{w^0_i w^1_i}} + 1946 \sin\left(\Pi(p^1_i/p^0_i)^{\sqrt{w^0_i w^1_i}} - \frac{p^1 \cdot q^1}{p^0 \cdot q^0} / \Pi(q^1_i/q^0_i)^{\frac{1}{2}(w^1_i + w^0_i)}\right)$$

This formula (as well as $f_2, \dots, f_{17}$) is exact in CD-case which may be thought to "rationalize" the use of (16) also in other situations. Of course, no such conclusions should be drawn.

The problem is that (6) determines the function f only in the subset $CD \subset \mathbb{R}^{4n}_{++}$ and we have given no rules to <u>extend its definition to $\mathbb{R}^{4n}_{++}$</u> from CD. Without such rules all above mentioned functions are appropriate extensions. By choosing functions g in (13) and K in (15) in various ways we can generate infinitely many extensions of $\Pi(p_i^1/p_i^0)^{w_i^0}$ from CD to $\mathbb{R}^{4n}_{++}$. And still more are easily invented. All these functions are usefull in the CD-world by (6) but this does not say anything of their usefulness in wider worlds, when $w_i^0 \neq w_i^1$. Completely other kind of tests (or properties) are required to judge if any of the functions we have mentioned or some other functions satisfying (6) are useful in wider contexts.

3. Examples, the CES-case

We have used the CD-case as an example because of its simplicity. But the conclusions are not limited to this simple case. In the same way we could generate an infinity of "exact index number formulas" e.g. in the CES(σ) case. In this Constant Elasticity of Substitution case, see Bergson (1936), Uzawa (1962) and Shephard (1970), the utility function has a representation

$$(17) \qquad u(q) = (\Sigma c_i q_i^{1-\frac{1}{\sigma}})^{1/(1-\frac{1}{\sigma})}$$

where c_i's are non-negative and CES-parameter $\sigma > 0$. The great majority of "exact index number formulas" in the CES(σ)-world would be useless in more general situations. A remarkable fact shown by Sato (1976) is that Vartia-Sato index (9), or

$$(18) \qquad VS\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = \Pi(p_i^1/p_i^0)^{L(w_i^1,w_i^0)/\Sigma L(w_j^1,w_j^0)}$$

is exact for all CES(σ)-worlds.

Therefore any function $f: \mathbb{R}^{4n}_{++} \to \mathbb{R}_{++}$ coinciding with (18) for all equilibrium points in the CES(σ)-world, or for points in

$$(19) \qquad CES(\sigma) = \{\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} \in \mathbb{R}^{4n}_{++} \mid (p^0,q^0) \text{ and } (p^1,q^1) \text{ are any two equilibrium points compatible with maximization of (17)}\}$$

We need to represent this set more explicitely. Using demand system $q = h(p,C)$ any equilibrium point (p,q) satisfies $q = h(p,p\cdot q)$ and conversely. Just like in the CD-case we have different representations for $h(p,C)$. A representation used e.g. by Lloyd (1975) is

$$(20) \qquad h_i(p,C) = \frac{(p_i/c_i)^{-\sigma}C}{(\Sigma c_j(p_j/c_j)^{1-\sigma})^{1/(1-\sigma)}} = \frac{(p_i/c_i)^{-\sigma}C}{e(p)}$$

where

$$(20b) \qquad e(p) = (\Sigma c_j(p_j/c_j)^{1-\sigma})^{1/(1-\sigma)} = (\Sigma c_j^{\sigma}p_j^{1-\sigma})^{1/(1-\sigma)}$$

is the price function, see Afriat (1972, p. 36). This is however an unillustrative representation. Let $(p^0,q^0)\in\mathbb{R}^{2n}_{++}$ be any fixed pair which satisfies (20): $q^0 = h(p^0,p^0\cdot q^0)$. Then we have

$$(21)\qquad \log\left(\frac{h_i(p,C)}{q_i^0}\right) = -\sigma\log\left(\frac{p_i/p_i^0}{(e(p)/e(p^0)}\right) + \log\left(\frac{C/C^0}{e(p)/e(p^0)}\right)$$

$$= -\sigma\log\left(\frac{p_i/p_i^0}{P(p,p^0)}\right) + \log\left(\frac{C/C^0}{P(p,p^0)}\right)$$

where $C^0 = p^0 \cdot q^0$ and $P(p,p^0)$ is the economic price index

$$(22)\qquad P(p,p^0) = e(p)/e(p^0)$$

$$= [\Sigma c_j(p_j/c_j)^{1-\sigma}/\Sigma c_j(p_j^0/c_j)^{1-\sigma}]^{1/(1-\sigma)}.$$

This however simplifies considerably when we choose an other parametrization by expressing the parameters $c_1,\ldots,c_n$ in terms of (p^0,q^0). Using the normalization

$$(23)\qquad \Sigma c_i = 1,\ u(q^0) = \left(\Sigma c_i(q_i^0)^{1-\frac{1}{\sigma}}\right)^{\frac{1}{1-1/\sigma}} = 1$$

and the first order conditions

$$(24)\qquad \partial u(q^0)/\partial q_i = \lambda p_i^0$$

we get after some easy manipulations

$$(25)\qquad c_i = p_i^0(q_i^0)^{1/\sigma}/p^0\cdot q^0$$

$$= (p_i^0 q_i^0/p^0\cdot q^0)\left(\frac{1}{q_i^0}\right)^{1-1/\sigma}$$

$$= w_i^0\left(\frac{1}{q_i^0}\right)^{1-1/\sigma}.$$

This gives c_i's in terms of 'old value shares' $w_i^0 = p_i^0 q_i^0 / p^0 \cdot q^0$ and 'old quantities' q_i^0. Inserting this to (17) gives

$$(26) \qquad u(q) = (\Sigma c_i q_i^{\,1-1/\sigma})^{\frac{1}{1-1/\sigma}}$$

$$= (\Sigma w_i^0 (q_i/q_i^0)^{1-\frac{1}{\sigma}})^{\frac{1}{1-1/\sigma}}$$

$$= (\Sigma w_i^0 (q_i/q_i^0)^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}.$$

This is a weighted moment mean (of order $\alpha = 1-1/\sigma$) of the quantity relatives (q_i/q_i^0), weighted by the old value shares. Note that no dimensional problems arise because value shares w_i^0 and quantity relatives (q_i/q_i^0) are dimensionless numbers, which are independent of all units of measurement. Therefore (26) seems to be the natural parametrization. Note also that $u(q) = 1$ for all q's indifferent to q^0, $q \sim q^0$ and $u(q) \gtrless 1$ according to $q \gtrless q^0$. Simularly we get for the price function

$$(27) \qquad e(p) = (\Sigma c_j^{\sigma}\, p_j^{1-\sigma})^{\frac{1}{1-\sigma}}$$

$$= (\Sigma (w_j^0)^{\sigma} (\frac{1}{q_j^0})^{\sigma-1}\, p_j^{1-\sigma})^{\frac{1}{1-\sigma}}$$

$$= (\Sigma (w_j^0)^{\sigma} (\frac{1}{p_j^0 q_j^0})^{\sigma-1} (p_j/p_j^0)^{1-\sigma})^{\frac{1}{1-\sigma}}$$

$$= (\Sigma (\frac{1}{p^0 \cdot q^0})^{\sigma-1}\, w_j^0 (p_j/p_j^0)^{1-\sigma})^{\frac{1}{1-\sigma}}$$

$$= p^0 \cdot q^0\, (\Sigma w_j^0 (p_j/p_j^0)^{1-\sigma})^{\frac{1}{1-\sigma}}.$$

Because $e(p^0) = p^0 \cdot q^0 = C^0$ we have for the price index (22)

(28) $$P(p,p^0) = e(p)/e(p^0)$$

$$= (\Sigma w_j^0 (p_j/p_j^0)^{1-\sigma})^{\frac{1}{1-\sigma}}.$$

This is also a weighted moment mean (but of order $\alpha = 1-\sigma$) of the price relatives (p_j/p_j^0), weighted by the old value shares $w_j^0 = p_j^0 q_j^0 / p^0 \cdot q^0$.

Actually $e(p)$ is the minimum expenditure needed to bye the utility determined by q^0 when prices are p. Therefore we could denote it more explicitely by $C(p,q^0)$, where $C(p,q^*)$ is the (minimum) expenditure needed to buy the utility level determined by q^* under prices p, or $C(p,q^*) = \min \{p \cdot q | u(q) \geq u(q^*)\}$. Afriat (1972) denotes this by $\rho(p,x)$ and calls it "utility cost function". He also shows that $C(p,q)$ factoririzes into price and quantity functions

(29) $$C(p,q^*) = e(p)u(q)$$

if and only if preferences are homothetic, or $q \sim q^* \Rightarrow kq \sim kq^*$ for all $k > 0$. The "antithetic price and quantity functions" in (29) are always linearly homogenous: $e(kp) = ke(p)$, $u(kq) = ku(q)$.

CES(σ)-preferences are homothetic and it may be checked using the Lagrangian technique with $F(q,\lambda) = p \cdot q - \lambda(u(q) - u(q^*))$ that

(30) $$C(p,q^*) = p^0 \cdot q^0 (\Sigma w_j^0 (p_j/p_j^0)^{1-\sigma})^{\frac{1}{1-\sigma}} (\Sigma w_j^0 (q_j^*/q_j^0)^{1-1/\sigma})^{\frac{1}{1-1/\sigma}}$$

$$= e(p)u(q^*),$$

in accordance with (26) and (27). We also have $C(p^0,q^0) = p^0 \cdot q^0$, $C(p,q^0) = e(p)$ and $C(p^0,q) = p^0 \cdot q^0\ u(q)$. Especially the price and quantity indices $P(p,p^0) = e(p)/e(p^0) = C(p,q^0)/C(p^0,q^0) = P(p,p^0;q^0)$, $Q(q,q^0) = u(q) = C(p^0,q)/C(p^0,q^0) = Q(q,q^0;p^0)$ satisfy

$$(31) \qquad \frac{C(p,q)}{C(p^0,q^0)} = P(p,p^0)Q(q,q^0)$$

for all p's and q's. As $C(p^0,q^0) = p^0.q^0$ equation (31) may be used to determine any of $C(p,q)$, $P(p,p^0)$, $Q(q,q^0)$ from other two. It is a "difficulty" in homothetic theory that identifying features (e.g. q^* in $P(p;\ p^0:q^*)$) cancel away and things become too "simple", or special. It is difficult to keep in mind what is given and what is derived.

Hicksian (or compensated) demand functions are derived by derivating the cost function (Shephard's theorem):

$$(32) \qquad H_i(p,q^*) = \partial C(p,q^*)/\partial p_i$$

$$= q_i^0(p_i/p_i^0)^{-\sigma}\ u(q^*)/P(p,p^0)^{-\sigma}.$$

This has beautiful representations in logarithms

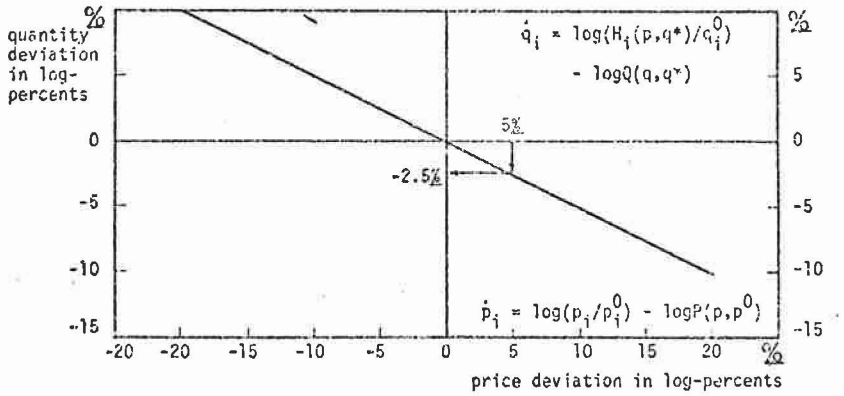
$$(33) \qquad \log\left(\frac{H_i(p,q^*)}{q_i^0}\right) = -\sigma \log\left(\frac{p_i/p_i^0}{P(p,p^0)}\right) + \log Q(q^*,q^0)$$

$$(34) \qquad \log\left(\frac{H_i(p,q^*)/q_i^0}{Q(q^*,q^0)}\right) = -\sigma \log\left(\frac{p_i/p_i^0}{P(p,p^0)}\right)$$

with numerous obvious interpretations, cf. also (21). According to (33) the log-change in the compensated demand $H^i(p,q^*)$ changes by the amount of the log-change in welfare and decreases in relation to the difference between the rate of change in its own price $\log(p_i/p_i^0)$ and the price level $\log P(p^1,p^0)$.

According to (34) the logarithmic deviation of an individual price ratio p_i/p_i^0 from the true price index $P(p,p^0)$, i.e. $\dot{p}_i = \log(p_i/p_i^0) - \log P(p,p^0)$, and the corresponding logarithmic deviation of the quantity ratio from the true quantity index $\dot{q}_i = \log(H_i(p,q^*)/q_i^0) - \log Q(q,q^*)$ depend on each other in an extremely simple way. They are negative multiples of each other, $\dot{q}_i = -\sigma\dot{p}_i$, the coefficient being the negative of the elasticity of the substitution σ, which is the same for all commodities a_i. We have used here the same notation as in Vartia (1978).

Figure 1. Relation between logarithmic deviations of price and quantity ratios from corresponding indices in CES(σ)-world



Suppose that σ = 0.5 and consider say "vegetables" a_i, whose prices have increased 10 % (read 10 log-percents, meaning $100 \log(p_i^1/p_i^0) = 10$) as the general price level has increased 5 % . The change in expenditure is 8 % and its real change 8 % - 5 % = 3 %. How much must the demand of vegetables change to fit the situation? We have from (33) $100 \log(q_i^1/q_i^0) = -0.5\ (10-5) + 3 = 0.5$, or the compensated demand of vegetables has to increase 0.5 %. Or in other words: If the prices of vegetables have increased 10 - 5 = 5 % more than average prices then its demand must "exceed" the growth of real expenditure by 0.5 - 3 = -2.5 % as shown in figure 1.

Those who like to think in terms of value shares may find the following expressions convenient. Let us define

the market value share functions

$$(35) \qquad w_i(p,C) = p_i h_i(p,C)/C$$

and the compensated value share functions

$$(36) \qquad w_i(p,q^*) = p_i H_i(p,q^*)/\sum_{j=1}^{n} p_j H_j(p,q^*)$$

$$= p_i H_i(p,q^*)/C(p,q^*).$$

The market and compensated value share systems are $w(p,C) = (w_1(p,C),\ldots,w_i(p,C))$ and $w(p,q^*) = (w_1(p,q^*),\ldots,w_n(p,q^*))$.

In CES (σ)-world these satisfy by (21) and (33):

$$(37) \qquad \log \frac{w_i(p,C)}{w_i(p^0,C^0)} = (1-\sigma) \log \frac{p_i/p_i^0}{P(p,p^0)}$$

$$(38) \qquad \log \frac{w_i(p,q^*)}{w_i(p^0,q^0)} = (1-\sigma) \log \frac{p_i/p_i^0}{P(p,p^0)} .$$

For $\sigma > 1$ the value share of the ith commodity decreases when its relative price increases; if $\sigma = 1$ the value shares are constant and we are back in the CD-case. Because all CES(σ)-systems are homothetic the value shares depend only on prices and are independent of income or standard of living, $w_i(p,C) = w_i(p,\bar{C})$ for all C and $\bar{C}$, and $w_i(p,q^*) = w_i(p,\bar{q})$ for all q^* and $\bar{q}$. Therefore there is no need to specify C or q^*

in (37) and (38). This is the curiosity of homothetic demand worlds. In fact $w(p,C) = w(p,q^*) = w(p)$ for all p,C and q^* in any homothetic demand world, i.e. the market and compensated value shares systems give same results if only their p's are the same. But they are different vector valued functions, because they are defined in different spaces.

Using the following representation for the log-change

$$(39) \qquad \log\left(\frac{x}{y}\right) = \frac{x-y}{L(x,y)}$$

where $L(x,y) = (x-y)/\log(x/y) \approx [2\sqrt{xy} + \frac{x+y}{2}]/3$ is <u>the logarithmic mean</u> of positive x and y, we get from (35) for any $w_i^1 = w_i(p^1,C^1)$ and $w_i^0 = w_i(p^0,C^0)$:

$$(40) \qquad w_i^1 - w_i^0 = (1-\sigma)\, L(w_i^1,w_i^0) \log \frac{p_i^1/p_i^0}{P(p^1,p^0)} \, .$$

This is valid for all i's and for all p's and w's in any CES(σ)-world. Therefore by summing

$$(41) \qquad \sum_{i=1}^{n} (w_i^1-w_i^0) = 1 - 1 = 0$$

$$= (1-\sigma) \sum_{i=1}^{n} L(w_i^1,w_i^0)[\log(p_i^1/p_i^0) - \log P(p^1,p^0)] \, ,$$

which is equivalent to

$$(42) \qquad \log P(p^1,p^0) = \sum_{i=1}^{n} \frac{L(w_i^1,w_i^0)}{\Sigma L(w_j^1,w_j^0)} \log\, (p_i^1/p_i^0).$$

This shows that Vartia-Sato index (9) is exact in all CES(σ)-worlds as shown first by Sato (1976).

Next we demonstrate some other index number results in CES(σ)-worlds. For $q^1 = h(p^1, C^1)$ we have

(43) $C^1 = C(p^1, q^1) = p^1 \cdot q^1$

and using (31) and $C^0 = p^0 . q^0$

(44) $p^1 \cdot q^1 / p^0 \cdot q^0 = P(p^1, p^0)\ Q(q^1, q^0)$.

This shows that for all equilibrium points (p^0, q^0) and (p^1, q^1), or points $(\frac{p^1}{p^0} \frac{q^1}{q^0})$ in CES(σ) the price and quantity indices (26) and (28) satisfy the weak factor reversal test, or multiply into the value ratio. From (43) e.g. $P(p^1, p^0)$ may be solved by dividing the value ratio by the quantity index

$$(45) \qquad P(p^1, p^0) = \frac{p^1 \cdot q^1}{p^0 \cdot q^0} \ / \ Q(q^1, q^0)$$

$$= \frac{p^1 \cdot q^1}{p^0 \cdot q^0} \ / \ \left(\Sigma w_i^0 (q_i^1 / q_i^0)^{\frac{\sigma - 1}{\sigma}}\right)^{\frac{\sigma}{\sigma - 1}}$$

This determines a new exact price index number formula in CES(σ)-world.

As e.g. Samuelson and Swamy (1974) show in any homothetic case e.g. the price index $P(p^1, p^0; q^*) = P(p^1, p^0)$ satisfies the time reversal test: $P(p^1, p^0) = 1/P(p^0, p^1)$. This gives us many new expressions for the price index $P(p^1, p^0)$. For instance from (28) and (45) we derive respectively

$$(46) \qquad f_{38}\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = 1/f_{28}\begin{pmatrix} p^0 & q^0 \\ p^1 & q^1 \end{pmatrix}$$

$$= 1/(\Sigma\, w_i^1 (p_i^0/p_i^1)^{1-\sigma})^{\frac{1}{1-\sigma}}$$

$$= (\Sigma\, w_i^1 (p_i^1/p_i^0)^{\sigma-1})^{\frac{1}{\sigma-1}}$$

$$(47) \qquad f_{39}\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = 1/f_{37}\begin{pmatrix} p^0 & q^1 \\ p^1 & q^0 \end{pmatrix}$$

$$= (\Sigma\, w_i^1 (q_i^0/q_i^1)^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} \Big/ \frac{p^0 \cdot q^0}{p^1 \cdot q^1}$$

$$= \frac{p^1 \cdot q^1}{p^0 \cdot q^0} \Big/ (\Sigma\, w_i^1 (q_i^1/q_i^0)^{\frac{1-\sigma}{\sigma}})^{\frac{\sigma}{1-\sigma}}$$

Usually (46) and (47) give different results but in the CES(σ)-world their difference vanishes, or

$$(48) \qquad f_{38}\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} - f_{39}\begin{pmatrix} p^1 & q^1 \\ p^0 & q^0 \end{pmatrix} = 0.$$

Therefore this "zero" (or any multiple of it) could be added anywhere say in (46) without disturbing its exactness in CES(σ)-world. But for instance adding $f_{38} - f_{39}$ to the denominator of (p_3^1/p_3^0) would spoil the formula in other situations. The reader may invent other dirty tricks which would leave exactness invariant.

As we noted in Vartia (1978, p. 21) in CES (σ=0) we have

(49) $\Sigma w_i^0(p_i^1/p_i^0) = p^1 \cdot q^0/p^0 \cdot q^0 = L$ (Laspeyres)

$= (\Sigma w_i^1(p_i^0/p_i^1))^{-1} = p^1 \cdot q^1/p^1 \cdot q^0 = P$ (Paasche)

$= \sqrt{\frac{p^1 \cdot q^0}{p^0 \cdot q^0} \frac{p^1 \cdot q^1}{p^0 \cdot q^1}} = \sqrt{LP} = F$ (Fisher)

$= \frac{1}{2}(L+P)$

$= \frac{p^1 \cdot (q^0+q^1)}{p^0 \cdot (q^0+q^1)} = \frac{L + C^1/C^0}{1 + (C^1/C^0)P}$ (Edgeworth)

Similarly in CES(σ=1) or CD case

(50) $\Pi(p_i^1/p_i^0)^{w_i^0} = \ell$ (Log-Laspeyres)

$= \Pi(p_i^1/p_i^0)^{w_i^1} = p$ (Log-Paasche)

$= \Pi(p_i^1/p_i^0)^{\frac{1}{2}(w_i^1+w_i^0)} = \sqrt{\ell p} = t$ (Törnqvist)

$= \Pi(p_i^1/p_i^0)^{\sqrt{w_i^1 w_i^0}/\Sigma\sqrt{w_j^1 w_j^0}}$ (Walsh)

$= \Pi(p_i^1/p_i^0)^{L(w_i^1,w_i^0)/L(w_j^1,w_j^0)}$ (Vartia-Sato)

to mention only some obvious cases. Note that all factor antithesis of the formulas e.g. $\frac{p^1 \cdot q^1}{p^0 \cdot q^0} / \Pi(q_i^1/q_i^0)^{w_i^0} = \frac{p^1 \cdot q^1}{p^0 \cdot q^0} / \Pi(q^1/q_i^0)^{w_i^1}$ etc. could be added here. These functions in (49)-(50) are all always reasonable index number formulas because of their <u>general properties</u>, which have nothing to do with their accidental identity in these special CES(σ)-cases. If again it happens that $\sigma = 2$ in the CES(σ)-world then

$$(51) \quad \Sigma w_i^1(p_i^1/p_i^0) = Pl \qquad \text{(Palgrave)}$$

$$= 1/\Sigma w_i^0(p_i^0/p_i^1) = Lh \qquad \text{(Harmonic Laspeyres)}$$

$$= \sqrt{w_i^1(p_i^1/p_i^0)/\ w_i^0(p_i^0/p_i^1)} = \sqrt{Pl \cdot Lh}$$

$$= \tfrac{1}{2}(Pl + Lh).$$

Also all the factor antithesis (FA for short) of the formulas give here same results so that e.g.

$$(52) \quad \Sigma w_i^1(p_i^1/p_i^0) = Pl \qquad \text{(Palgrave)}$$

$$= \frac{p^1 \cdot q^1}{p^0 \cdot q^0} / \Sigma\ w_i^1(q_i^1/q_i^0) \qquad \text{(FA of Palgrave)}$$

$$= \left(\frac{p^1 \cdot q^1}{p^0 \cdot q^0}\right) / \Sigma w_i^0(q_i^0/q_i^1) \qquad \text{(FA of Lh)}$$

These curious identities of CES(σ=2)-world are "in practice" usually far from being satisfied, see Vartia (1978).

Less evident identities in <u>all</u> CES(σ)-cases are

$$[\Sigma w_i^0(p_i^1/p_i^0)^{r/2}/\Sigma w_i^1(p_i^1(p_i^1/p_i^0)^{-r/2}]^{1/r} \tag{53}$$ (Quadratic mean of order r)

$$= \frac{p^1 \cdot q^1}{p^0 \cdot q^0} \Big/ [\Sigma w_i^0(q_i^1/q_i^0)^{r/2}/\Sigma w_i^1(q_i^1/q_i^0)^{-r/2}]^{1/r}$$ (FA of above)

$$= \Pi(p_i^1/p_i^0)^{L(w_i^1,w_i^0)/\Sigma L(w_j^1,w_j^0)}$$ (Vartia-Sato)

with $r = 1-\sigma$. Quadratic means of order r are considered e.g. by Denny (1974), Diewert (1976, 1978), Sato (1974) and Vartia (1978) and shown to be useful index number formulas by many independent criterions. Note that there are no parameters to be estimated in Vartia-Sato index so that it adjusts itself automatically. In quadratic means above the right $r = 1-\sigma$ must be known beforehand.

4. Conclusion

We have demonstrated that a great many different functions f: $\mathbb{R}_{++}^{4n} \to \mathbb{R}_{++}$ which may or may not deserve the name "index number formula" can be constructed so that they are "exact" in CD or CES worlds. Some of these functions are useful elsewhere, most are not. The problems do not limit to these special demand systems but arise similarly in any demand or production system. This calls for a systematic investigation of the properties we would demand or desire of an index number formula; i.e. an axiomatic treatment of index number formulas in the spirit of Fisher (1922), Vartia (1976) and Eichhorn & Voeller (1977).

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