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RENEWABLE RESOURCES IN AN OVERLAPPING GENERATIONS ECONOMY WITHOUT CAPITAL****

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ABSTRACT: We incorporate a renewable resource into an overlapping generations model without capital and with quasi-linear preferences. Besides being an input for production the resource serves as a store of value. We characterize the dynamics, efficiency and stability of the steady state equilibria. The stability properties are sensitive to the type of the resource growth. For constant growth there is only one steady state equilibrium which is stable and efficient. In the general case of the concave growth function there are usually at least two steady state equilibria, one of which is stable and the other one unstable. The unstable steady state is efficient, but the stable one may or may not be. We study the robustness of our results by assuming a logarithmic periodic utility function. We show that for the Cobb-Douglas production function the steady state is unique and stable regardless of whether the equilibrium is efficient or inefficient. Our analytical results are illustrated by numerical calculations.

Keywords: overlapping generations, renewable resources.

JEL classification: D90, Q20.

1. INTRODUCTION

Traditional theories of renewable resource use assume an infinitely lived agent or a social planner, and have the property that there is one steady state equilibrium, which is a saddle. Equilibrium is a function of resource demand (price), costs and exogenous real interest rate (for economics of forestry and fisheries, see e.g. [6] and [7]). These models do not account for the fact that in practice renewable resources are important stores of value between different generations.[1] Hence, one can ask whether this analysis is robust in an overlapping generations economy, where agents have a finite life but resource stock may grow forever, and where the real interest rate is determined endogenously.

Recent studies ([9], [11], [13] and [14]) focusing on the sustainable use of renewable resources within the overlapping generations framework have provided a partial answer.[2] They establish the generally well-known fact that competitive overlapping generations economies may be inefficient. [9] demonstrates that a competitive economy with constant population may under-harvest its renewable resources as a consequence of the resource being inessential for production. In a different vein, [14] shows that both a low rate of resource regeneration relative to population growth and a low level of saving may lead to unsustainable use of renewable resources, so that consumption declines over time.

These papers study the steady state equilibrium without analyzing its dynamics and stability. This is an unfortunate drawback, since stability properties of the renewable resource exploitation are important e.g. in policy making. If the utilization of the resource tends to be unstable, competition may more easily lead to the destruction of the whole resource, which naturally necessitates a more careful resource management.

In this paper we characterize the steady state equilibrium of a general equilibrium overlapping generations economy, study its stability properties, and compare competitive and efficient steady state equilibria. For this purpose, we construct a model where agents live for two periods. The renewable resource serves both as a store of value and as an input in the production of consumption good.

Our focus is entirely on the extractive use of resource and we omit amenity services provided by the resource. The resource stock may be interpreted, for instance, as forests or fisheries (with well-defined property rights over fishing stocks). Unlike [9] and [14], who assume constant and linear growth, we utilize a general concave resource growth function, which captures the essential features of renewable resources more adequately. As a special case we analyze also the use of expendables, for which the growth rate is independent of the resource stock (relevant e.g. for the use of hydropower or most agricultural production, see [18] for the resource classification). Our model is similar to the exhaustible resource model of Olson and Knapp, see [15].

It will turn out that for a model with quasi-linear utility function the type of growth function plays a very important role in the analysis. Under constant growth there is one

1 Tobin , for instance, points out that "land and durable goods, or claims upon them are principal stores of value" [19, p. 83].

2 In addition to the above references in the OLG framework, see e.g. [1] for an analysis of the effects of forest and inheritance taxation on harvesting, stand investment and timber bequests in an overlapping generations model with one-sided altruism. For pollution as an intergenerational externality, see [8].

equilibrium which is stable, indicating that the overlapping generations economy does not qualitatively differ from the world of infinitely living agents of traditional renewable resource theories. Assuming concave resource growth, however, brings a striking difference to the results of traditional analyses. Instead of one equilibrium, there are usually at least two steady state equilibria, one of which is stable and the other one unstable.

To explore the robustness of our results with quasi-linear utility, we also impose concavity on both periodic utility functions via logarithmic specification. Now the dynamical system reduces to a non-linear first-order difference equation for the resource stock (harvesting being determined recursively). We show that for the Cobb-Douglas production function the steady state equilibrium is unique and stable regardless of whether the equilibrium is efficient or inefficient, and irrespective of the type of the growth function. Since the steady state equilibrium is determinate, the qualitative properties of the model under logarithmic utility function are similar to the saddle point equilibrium.

The paper is organized as follows. In section 2 the basic structure of the model is developed. Section 3 analyzes steady state equilibria, while dynamical equilibria are studied in section 4, and efficiency of competitive equilibrium in section 5. Dynamical equilibria with logarithmic utility are examined in section 6. Numerical calculations with parametric specifications are presented in section 7. This is followed by a concluding discussion.

2. THE MODEL AND THE EQUILIBRIUM CONDITIONS

We consider an overlapping generations economy without population growth, where agents live for two periods. We assume that agents maximize the intertemporally additive, quasi-linear lifetime utility function

$$V = u(c_1^t) + \beta c_2^t, \tag{1}$$

where c_i^t denotes the period i (=1,2) consumption of consumer-worker born at time t and $\beta = (1+\delta)^{-1}$ with δ being the rate of time preference.[3] In addition to simplifying the analysis, quasi-linear specification allows us to focus more sharply on the importance of the time preference for the use of a renewable resource. We assume for the first period utility function that $u' > 0$, $u'' < 0$ and $\lim_{c \to \infty} u'(c) = 0$ and $\lim_{c \to 0} u'(c) = \infty$. The young are endowed with one unit of labor, which they supply inelastically to firms in consumption goods sector. The labor earns a competitive wage. The representative consumer-worker uses the wage to buy consumption good and save in the financial asset or to buy the available stock of the renewable resource.

[3] Assumption of quasi-linearity produces a saving function with positive interest rate elasticity.

The firms in the consumption good sector have a constant returns to scale technology, $F(H_t, L_t)$, to transform the harvested resource (H_t) and labor (L_t) into output. This technology can be expressed in factor intensive form to give $F(H_t, L_t)/L_t = f(h_t)$ with the standard properties $f' > 0$ and $f'' < 0$. Furthermore, we assume that $\lim_{h \to 0} f'(h_t) = \infty$ and $\lim_{h \to \infty} f'(h_t) = 0$, where h_t $(= H_t / L_t)$ is the per capita level of harvest.

The growth of the renewable resource is $g(x_t)$, where x_t denotes the beginning of period t stock of the resource. $g(x_t)$ is assumed to be a strictly concave function, i.e. $g'' < 0$. Furthermore, we assume that there are two values $x = 0$ and $x = \tilde{x}$ for which $g(0) = g(\tilde{x}) = 0$. Consequently**,** there is a unique value $\hat{x}$ at which $g'(\hat{x}) = 0$ where $\hat{x}$ denotes the stock providing the maximum sustained yield (MSY), and $\tilde{x}$ is the stock at which growth is zero. It is the maximal stock that the natural environment can sustain. For instance, a logistic growth function ($g(x) = ax - (1/2)bx^2$) fulfills these assumptions.

The renewable resource in our model has two roles. It is both a store of value and an input in the production of consumption good. The market for the resource operates in the following manner. At the beginning of the period the old agents own the stock, and also receive that period's growth of the stock. They sell the stock (growth included) to the firms, which then decide how much of that resource to harvest and use as an input in the production of the consumption good. The firm will sell the remaining stock of the resource to the young at the end of the period. Besides owning the stock the current old generation (generation t-1 in period t) will also get its growth, so that the stock they have available for trading is $x_t + g(x_t)$.

It is interesting to note that via growth function this "natural" production activity yields a profit for its owner. These profits could presumably be distributed in alternative ways. For instance, there could be a stock market where the ownership rights for the resource are exchanged. The young buy the shares for the resource, and when old, get the dividend and the proceeds from selling the shares next period.[4] This kind of arrangement leads to the same allocation, which we will have in our model.[5]

The transition equation for the resource is

$$x_{t+1} = x_t - h_t + g(x_t), \tag{2}$$

where h_t denotes that part of the resource stock which has been harvested for use as an input in production. The initial stock and its growth, $g(x_t)$, can be conserved for the next period's stock or used for this period's harvest.

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[4] We are thinking here about the stock market arrangements proposed by Lucas and Brock (see, [10] [3] and [4]). Since [10] and [2] have infinitely lived agents, the treatment of the stock market in their papers cannot readily be applied to our overlapping generations model, where e.g. there is limited market participation. [3] presents an overlapping generations version of the asset pricing model of [10], where the asset pays a constant dividend each period. For a recent treatment of the stock market in an overlapping generations model with capital, see [12].

[5] A sketch of the proof is available from the authors upon request.

In addition to trading in the resource market, the young can also participate in the financial markets by borrowing or lending, the amount of which is denoted by s_t. The periodic budget constraints are thus

$$c_1^t + p_t x_{t+1} + s_t = w_t \tag{3}$$

$$c_2^t = p_{t+1}\left[x_{t+1} + g(x_{t+1})\right] + R_{t+1} s_t \tag{4}$$

where p_t is the price of the resource stock in terms of period t's consumption, w_t is the wage rate, and $R_{t+1} = 1 + r_{t+1}$ is the interest factor. The young generation buys an amount x_{t+1} of the resource stock from the representative firm. The firm harvests an amount h_t of the stock, and x_{t+1} is left to grow. According to (4) the old generation consumes their savings including the interest, and the income they get from selling the resource next period to the firm, $p_{t+1}\left[x_{t+1} + g(x_{t+1})\right]$.

The periodic budget constraints (3) and (4) imply the lifetime budget constraint

$$c_1^t + \frac{c_2^t}{R_{t+1}} = w_t + \frac{p_{t+1}\left[x_{t+1} + g(x_{t+1})\right] - R_{t+1} p_t x_{t+1}}{R_{t+1}}. \tag{5}$$

Maximizing (1) subject to (5) and the appropriate non-negativity constraints gives the following first-order conditions for s_t and x_{t+1} at the interior solution

$$u'(c_1^t) = R_{t+1}\beta \tag{6}$$

$$p_t u'(c_1^t) = p_{t+1}\left[1 + g'(x_{t+1})\right]\beta . \tag{7}$$

Due to the linear second period utility function we might get corner solutions (i.e. $c_2^t = 0$) for some parameter values. This happens, for instance, if β is low enough, implying that the consumer does not want to consume anything in the second period.[6]

These conditions have straightforward interpretations. The Euler equation (6) says that the marginal rate of substitution between today's and tomorrow's consumption should be equal to the interest factor. According to (7) the marginal rate of substitution between consumptions in two periods should be equal to the resource price adjusted growth factor. (6) and (7) together imply the arbitrage condition for two assets

[6] To make the left-hand side of (6) very small for a given R_{t+1}, the first-period consumption must be very large. This is not, however, possible since the level of the stock, harvest and consumption in any period are bounded. We elaborate the issue of corner solution later when discussing the existence of steady state equilibrium.

$$R_{t+1} = [1 + g'(x_{t+1})]\frac{p_{t+1}}{p_t}, \tag{8}$$

which says that the interest factor is equal to the resource price adjusted growth factor. Using (8) we can rewrite the lifetime budget constraint as

$$c_1^t + \frac{c_2^t}{R_{t+1}} = w_t + \frac{p_{t+1}[g(x_{t+1}) - g'(x_{t+1})x_{t+1}]}{R_{t+1}}, \tag{9}$$

where the term in the square brackets is positive.

We next characterize the equilibria and dynamics of the model. The competitive equilibrium is defined as follows.

Definition. A sequence of a price system and a feasible allocation,

$\{p_t, R_t, w_t, c_1^t, c_2^{t-1}, h_t, x_t\}_{t=1}^{\infty}$ is a competitive equilibrium, if

(i) given the price system consumers maximize subject to their budget constraints

and

(ii) markets clear for all t = 1,2,...,T,...

Market clearing conditions are

$$c_1^t + c_2^{t-1} = f(h_t) \tag{10a}$$

$$x_{t+1} + h_t = x_t + g(x_t) \tag{10b}$$

$$s_t = 0 \tag{10c}$$

$$f'(h_t) = p_t \tag{10d}$$

$$f(h_t) - h_t f'(h_t) = w_t \tag{10e}$$

Equation (10a) is the resource constraint for all t, and (10b) is the transition equation for the renewable resource stock. The fact that there is only one type of a consumer per generation and no government debt forces the asset market clearing condition to be such that saving is zero for all t. Equations (10d) and (10e) in turn are the first-order conditions for profit maximization, and determine the evolution of prices, p_t and w_t.

Market clearing condition (10b) and the first-order condition (7) for the resource stock and harvesting imply the following planar system that describes the dynamics of the model

$$x_{t+1} = x_t - h_t + g(x_t) \tag{11}$$

$$f'(h_t)u'[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}] = \beta f'(h_{t+1})[1 + g'(x_{t+1})], \tag{12}$$

where we have used the periodic budget constraints (3) and (4), and the equilibrium conditions (10d) and (10e), to arrive at the equilibrium Euler equation (12). Equations (11) and (12) are the main objects of our study. Before analyzing the dynamic properties of this system we characterize the steady state equilibrium.

3. STEADY STATE EQUILIBRIA

The planar system describing the dynamics of the resource stock and harvesting consists of equations (11) and (12). The steady states $\Delta x_t = \Delta h_t = 0$ fulfill the following equations

$$h = g(x) \tag{13}$$

$$u'\left[f(h) - f'(h)(h+x)\right] = \beta\left[1 + g'(x)\right], \tag{14}$$

and define two stationary loci in the hx - space for which the resource stock and harvesting remain unchanged. Total differentiation of (13) and (14) yields

$$\left.\frac{dh}{dx}\right|_{(\Delta x=0,\Delta h=0)} = g' \tag{15}$$

$$\left.\frac{dh}{dx}\right|_{(\Delta h=0,\Delta x=0)} = \frac{u'' f' + \beta g''}{-u'' f''(h+x)} > 0\,. \tag{16}$$

These equations describe the slopes of the curves $\Delta x_t = \Delta h_t = 0$ in the hx- space. While the slope of the curve defined by (15) is not monotone, by (16) the steady state Euler equation is an increasing curve in the hx-space.

The curve (14) must lie above the curve $c_1 = f(h) - f'(h)(h+x) = 0$, since the first-period consumption must be positive due to the condition $\lim_{c \to 0} u'(c) = \infty$. To get further insight on how this requirement affects the number of steady states (and also on the possible dynamic paths) we consider the inequality $c_1 > 0 \Leftrightarrow f(h) - f'(h)(h+x) > 0$. The lower bound on positive consumption can be rearranged to obtain

$$c_1 = f\left[1 - \frac{f'h}{f}\right] - f'x, \tag{17}$$

where $0 < f^{-1} f'h = 1/\alpha < 1$ is the elasticity of output with respect to harvest, i.e. the factor share of the renewable resource in production.[7] It is straightforward to show that the curve

[7] We can rewrite this condition as $c_1 > 0 \Leftrightarrow (h/x) > \alpha/(1-\alpha)$, which is analogous to the respective condition developed by Olson and Knapp [15, pp. 281–282] for the model with exhaustible resource.

$f(h)-f'(h)(h+x)=0$ is upward sloping in hx-space. The following lemma establishes that the curve goes through the origin.

Lemma 1. *The point* $\{h=0, x=0\}$ *fulfills the curve* $f(h)-f'(h)(h+x)=0$.

Proof. See Appendix 1.

This is natural since there can be no consumption if there is no resource and harvesting (a necessary input in production). Next we study the behavior of the Euler equation in a steady state

$$u'[f(h)-f'(h)h-f'(h)x]=\beta[1+g'(x)]. \tag{18}$$

Hence, we have

Lemma 2. *A point* $\{h>0, x=0\}$ *fulfills* $u'[f(h)-f'(h)h-f'(h)x]=\beta[1+g'(x)]$.

Proof. See Appendix 1.

To study the existence of the steady state we rewrite the Euler equation (14) by using $h=g(x)$ as follows

$$LHS(x)\equiv u'[f(g(x))-f'(g(x))g(x)-f'(g(x))x]=\beta[1+g'(x)]\equiv RHS(x). \tag{19}$$

Differentiation of both sides of (19) gives

$$RHS'(x)=\beta\, g''(x)<0 \tag{20}$$

$$LHS'(x)=-u''(c_1)[f'+f''g'(x+g(x))]=? \tag{21}$$

As for the limiting behavior of $LHS(x)$, recall that $\tilde{x}$ is such that $g(\tilde{x})=0$. If $x\to\tilde{x}$ the argument of the marginal utility approaches minus infinity because of the Inada condition for the production function. Hence, there must be a value of x, say $x'<\tilde{x}$, such that this argument approaches zero. This means that the marginal utility (i.e. the value of $LHS(x)$) approaches infinity. On the other hand, if $x\to 0$, the first period consumption approaches zero and the marginal utility approaches infinity. It is also clear that the argument of the marginal utility cannot reach the value infinity for any $0<x<\tilde{x}$, so that the function $LHS(x)$ cannot touch the x-axis on that interval. Because $LHS'(x)=0$ is equivalent of having $f'+f''g'[x+g(x)]=0$, it means that the minimizing level of the stock is such that the first-period steady state consumption ($f[g(x)]-f'[g(x)]g(x)-f'[g(x)]x$) is maximized. We denote that level of x as x_c and the corresponding consumption as c_{1m}.

Since there is no Inada condition for the growth function, $\beta[1+g'(0)]$ is just a finite number. Lowering the value of the discount factor (i.e. β) tilts the function $RHS(x)$ downwards making its slope less steep. Hence, there must be a lower bound for the discount factor, say $\underline{\beta}$, such that for any $\beta<\underline{\beta}$, the steady state equilibrium does not exist. We have drawn the case with two nontrivial steady states in Figure 1. Since the function $LHS(x)$ does not depend on the value of β , we can lower the function $RHS(x)$ so much that the two curves neither cross nor touch.

For the nonexistence of equilibrium (see Figure 1) at least the following necessary, but not sufficient condition (the $RHS(x)$ curve must lie below the $LHS(x)$ curve at the minimum point of the $LHS(x)$), must hold for the discount factor $u'(c_{1m})>\beta[1+g'(x_c)]$, where $c_{1m}=f[g(x_c)]-f'[g(x_c)]g(x_c)-f'[g(x_c)]x_c$. It can be rewritten as

$$\beta<\frac{u'(c_{1m})}{1+g'(x_c)}. \tag{22}$$

Thus it is necessary for the nonexistence of the steady state that the discount factor must be lower than the quantity on the right-hand side, which in turn depends on the identifiable and economically meaningful allocation, i.e. there is an upper bound for this lower bound.[8] This nonexistence result follows from the fact that there is no Inada condition for the second period linear utility function. As discussed in section 2, this means that there can be circumstances (e.g. a very low discount factor), when consumers do not want to consume anything next period.

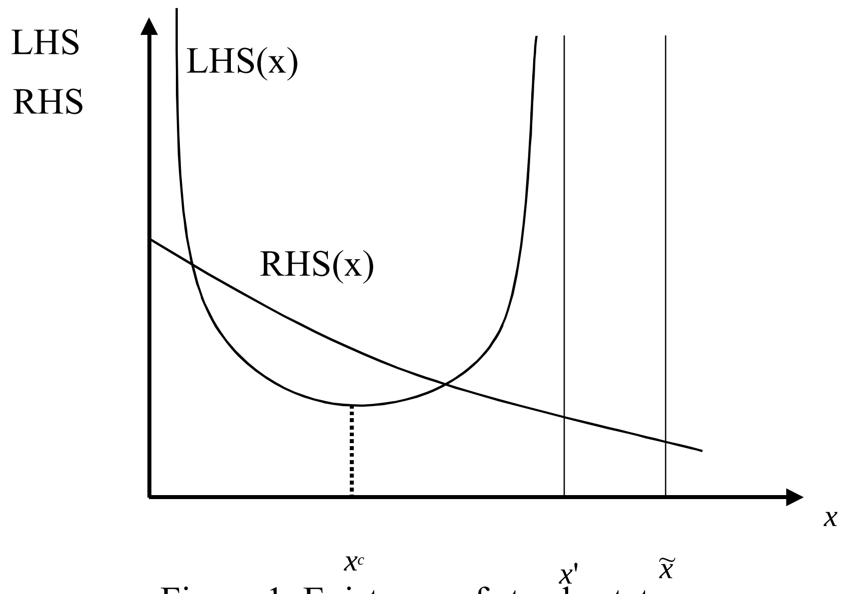

Figure 1. Existence of steady states

[8] For the more general role of the bounds of the discount factor for conservation versus extinction of a renewable resource, see [16].

The preceding discussion has made it clear that the steady state in our model is not necessarily unique. Given that the steady state Euler equation starts at the point $\{h > 0, x = 0\}$, if there are steady state equilibria, there are at least two of them, except for the rare case, where the Euler equation and the growth curve are tangent to each other. When the growth rate, $g'(x)$, is positive, the upward sloping Euler equation can cross the growth curve in many points. For two steady states it is necessary that the Euler equation cuts the growth curve first from above and then from below. On the portion of the growth curve where $g'(x) \leq 0$ there can be only one steady state equilibrium.

We will describe the loci $\Delta x_t = 0$ and $\Delta h_t = 0$ in the hx-space by totally differentiating (13) and (14). The slope of the locus, $h_t = g(x_t)$, when $\Delta x_t = 0$ (but h may vary) and evaluated at the steady state is

$$\left.\frac{dh_t}{dx_t}\right|_{\Delta x_t = 0} = g'(x) \,. \tag{23}$$

The slope of the Euler equation, when $\Delta h_t = 0$ (but x may vary), and evaluated at the steady state is

$$\left.\frac{dh_t}{dx_t}\right|_{\Delta h_t = 0} = \frac{(u'' f' + \beta g'')(1 + g')}{u''[f' - f''(x+h)] + \beta g''} > 0 \,. \tag{24}$$

While the slope in (23) can be positive, zero or negative, the slope in (24) is always positive given our assumptions on the utility function and the fact that $1 + g'$ >0, because in the steady state equilibrium $1 + g'$ equals the interest factor (c.f. arbitrage equation (8)).

We collect the previous discussion in

Proposition 1. *If the discount factor, β, is "low enough", the steady state may not exist. If the steady state exists, there are at least two of them, except for the rare case where the Euler equation and the growth curve are tangential to each other.*

When β is "low enough", the economy consumes the entire resource stock despite its capability of providing new stock via growth, so that resource use is not sustainable. This extinction result derives from the combination of quasi-linearity and zero harvest costs in our model, while e.g. in some traditional fisheries models the harvest costs increase with the decrease of the stock, preventing extinction.

In what follows we concentrate on the case of two steady states, i.e., the Euler equation cut the growth curve from below in equilibrium with the larger level of resource stock (see Figures 2 and 3 below).

$$\left.\frac{dh_t}{dx_t}\right|_{\Delta h_t = 0} > \left.\frac{dh_t}{dx_t}\right|_{\Delta x_t = 0} . \tag{25}$$

4. DYNAMICAL EQUILIBRIA

To study the dynamics of our model we start by considering paths for which $x_{t+1} \geq x_t$ and $h_{t+1} \geq h_t$. It follows from (11)

$$x_{t+1} \geq x_t \Leftrightarrow x_t - h_t + g(x_t) \geq x_t \quad \Leftrightarrow g(x_t) \geq h_t, \tag{26}$$

and from (12)

$$h_{t+1} \geq h_t \Leftrightarrow f'(h_{t+1}) \leq f'(h_t) \Leftrightarrow \frac{u'\left[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}\right]}{\beta\left[1 + g'(x_{t+1})\right]} \leq 1. \tag{27}$$

Equations (26) and (27) represent the area in the state space where the variables x and h are weakly increasing and also the complementary area in which they are strictly decreasing.

We will rewrite equations (11) and (12) as follows

$$x_{t+1} = x_t - h_t + g(x_t) \equiv G(x_t, h_t) \tag{28}$$

$$f'(h_{t+1}) = \frac{f'(h_t)u'\left[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}\right]}{\beta\left[1 + g'(x_{t+1})\right]}. \tag{29}$$

Substituting the RHS of (28) for x_{t+1} in (29) gives an implicit equation for h_{t+1},

$$h_{t+1} = F(x_t, h_t). \tag{30}$$

The planar system describing the dynamics of the resource stock and harvesting consists now of equations (28) and (30). The Jacobian matrix of the partial derivatives of the system is

$$\mathrm{J} = \begin{bmatrix} G_x & G_h \\ F_x & F_h \end{bmatrix},$$

where partial derivatives can be calculated (and evaluated at the steady state) as

$$G_x(x,h) = 1 + g' \qquad\qquad G_h(x,h) = -1$$

$$F_x(x,h) = \frac{-(f')^2 u'' - f'\beta g''}{\beta f''} < 0$$

$$F_h(x,h) = 1 + \frac{(f')^2 u''}{(1+g')\beta f''} - \frac{f'u''(x+h)}{(1+g')\beta} + \frac{f'g''}{(1+g')f''} > 0.$$

Based on these partial derivatives, the trace T and the determinant D of the characteristic polynomial can be calculated to be

$$D = 1 + g' - \frac{f'u''(x+h)}{\beta} > 0,$$

$$T = 2 + g' + \frac{(f')^2 u''}{(1+g')\beta f''} - \frac{f'u''(x+h)}{(1+g')\beta} + \frac{f'g''}{(1+g')f''} > 1.$$

It is easy to see that $D + T + 1 > 0$ holds. The nature of the stability of the steady state depends then crucially on the sign of $D - T + 1$. In determining this sign we use information about the behavior of Euler equation and the growth curve at both steady states (see Appendix 1 for details). Armed with these calculations, we get

Proposition 2. *In the case of concave resource growth with two steady states, the one associated with a larger natural resource stock is stable (a saddle), while the other with a smaller stock is unstable (a source).*

Proof: See Appendix 2.

Figures 2 and 3 describe the dynamics in the case of two steady states. In Figure 2 the larger steady state equilibrium stock lies on the right-hand side, while in Figure 3 on the left-hand side of the maximum sustained yield $\hat{x}$. The steady state is at the maximum sustained yield only accidentally. The equilibrium with the smaller stock is unstable and with the larger one is stable.

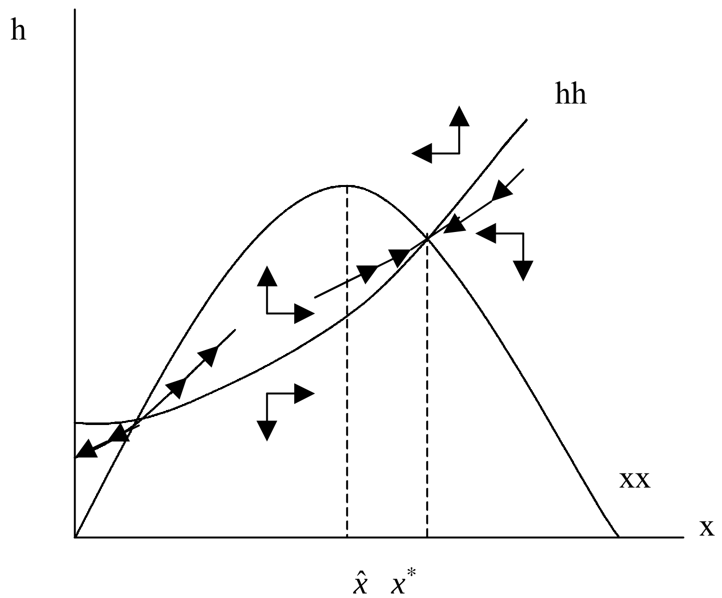

Figure 2. $x^* > \hat{x}$

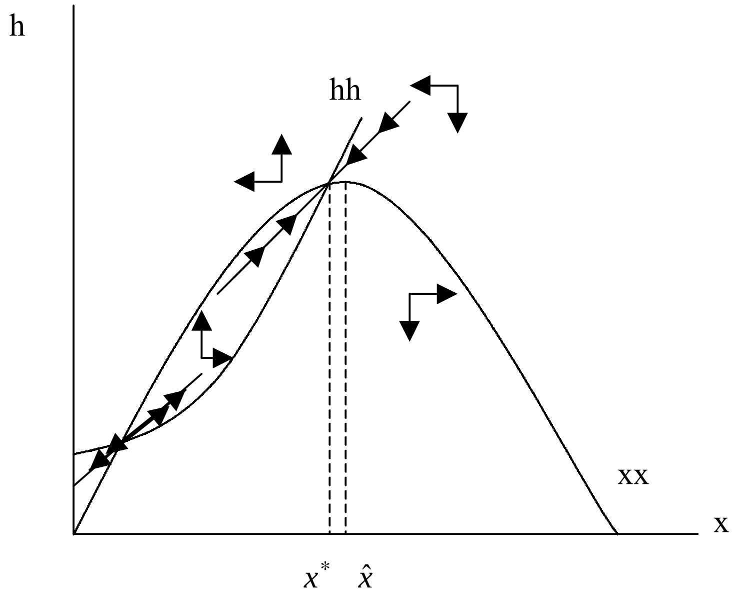

Figure 3. $x^* < \hat{x}$

We will next study briefly the case of expendables where the growth function is constant so that $g(x) = m > 0$ as in [9], [13] and [14]. Then the dynamics in (11) and (12) will be modified to

$$x_{t+1} \geq x_t \Leftrightarrow x_t - h_t + m \geq x_t \quad \Leftrightarrow m \geq h_t \tag{31}$$

$$h_{t+1} \geq h_t \Leftrightarrow f'(h_{t+1}) \leq f'(h_t) \Leftrightarrow u'\left[f(h_t) - f'(h_t)(h_t + x_t)\right] \leq \beta\,, \tag{32}$$

and the steady state is characterized by the following equations

$$m = h \tag{33}$$

$$u'\left[f(h) - f'(h)(h + x)\right] = \beta\,. \tag{34}$$

For the existence of a nontrivial steady state we need to assume that $u'\left[f(m) - f'(m)m\right] < \beta$. This means that *hh*-phaseline starts below *m*, which is clearly the case for high enough *m*. Total differentiation of (34) yields

$$\frac{dh}{dx}|(\Delta h = 0) = \frac{f'}{-f''(h + x)} > 0\,. \tag{35}$$

Figure 4 describes the resulting dynamics. With constant growth we get at most only one steady state at the point where the upward-sloping *hh*-phaseline crosses the constant growth *xx*-phaseline. This steady state is stable (a saddle).[9]

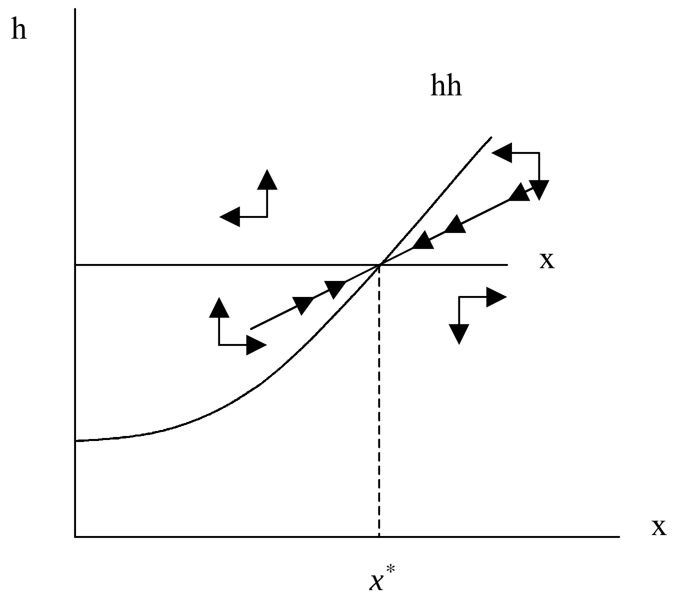

Figure 4. **Constant growth**

Thus we have,

Corollary 1. *In the case of constant growth there is a unique steady state which is stable.*

Proposition 2 and Corollary 1 reveal how, in the case of quasi-linear preferences, the nature of the growth function matters both for the number of steady states and the stability of the system. The difference to the results of traditional renewable resource theories is striking. While they have only one stable steady state, assuming general concave resource growth in the overlapping generations economy produces usually at least two steady states, one of which is stable and the other unstable.

5. EFFICIENCY OF STEADY STATE EQUILIBRIA

To investigate the efficiency of steady state equilibria, we explicitly take into account the welfare of the oldest generation, and denote the weight of its utility function in the social

9 This can be seen from the proof in Appendix 1 by setting $g = m$ so that $g' = g'' = 0$.

welfare function by $\phi > 0$. The efficient equilibrium is obtained by solving the following social planner's problem

$$\max_{\{c_2^0, c_1, c_2, h, x\}} W = \phi c_2^0 + u(c_1) + \beta c_2 \tag{PE}$$

s.t. $h = g(x)$

$c_1 + c_2 = f(h)$

$c_1 + c_2^0 = f(h)$

$x + h = x_1 + g(x_1)$,

where x_1 is the initial stock of the resource owned by the initial old generation. In Appendix 3 we show that efficient equilibria are characterized by the condition $R = 1 + g'(x) \geq 1$. Hence, all the equilibria for which $g'(x) \geq 0$ are efficient, while those with $g'(x) < 0$, are inefficient. Equilibria with $g'(x) < 0$ are inefficient because consumption could be increased for every generation by harvesting some of the stock. But equilibria with $g'(x) > 0$ are efficient, because trying to increase the stock to the maximum level will force the consumption of some generation to be lowered.[10] If the weight of the oldest generation were zero, we would obtain $g'(x) = 0$ for efficiency, which defines the maximum sustained yield (MSY) stock.

How do these findings relate to the properties of steady states in standard overlapping generations models? Given the arbitrage condition (8), the real rate of interest equals $g'(x)$ in the steady state. The case $g'(x) < 0$ corresponds to the situation where the real interest rate is less than the population growth rate (zero in our model), and the natural resource has been overaccumulated. $g'(x) > 0$ corresponds the case where the real interest rate exceeds population growth rate, and thus is efficient.[11]

Inefficiency in our model results from the overlapping generations structure. Unlike in models, where the first fundamental theorem of welfare economics holds, there is a double infinity of consumers and dated commodities (consumptions in each period) in an overlapping generations model. As pointed out by [17] this double infinity (and not the limited market participation) is the fundamental reason for inefficiency in overlapping generations models.

---

[10] This can be seen as follows. Consider e.g. some period τ where stock is increased. Up to that period the economy has been in a steady state where h = g(x). So in period τ $h_\tau < g(x_\tau)$. This means first that $x_{\tau+1} = x_\tau - h_\tau + g(x_\tau)$, where now obviously $x_{\tau+1} > x_\tau$. Furthermore, because $h_\tau < h$ we have $f(h_\tau) < f(h)$, which means that consumption is decreased at least for one generation. Later generations will get higher consumption because the stock has increased.

[11] Efficiency outside steady states is a more complicated problem. One can study the efficiency of these paths by modifying the criterion developed by [5].

We summarize the discussion above in

Proposition 3. *In the case of concave growth with two steady states, the unstable one is always efficient but the stable one may or may not be efficient.*

It is also easy to see that

Corollary 2. *In the case of constant growth the steady state is efficient.*

6. DYNAMICAL EQUILIBRIA WITH LOGARITHMIC UTILITY

To explore the robustness of our results with quasi-linear utility, we relax the assumption of the linear second period utility function, but maintain the assumption of a general concave resource growth. Specifically, we consider a case, where the both periodic utility functions are logarithmic so that the intertemporal elasticity of substitution is unity. In this case (12) can be written as

$$\frac{f'(h_t)}{f(h_t)-h_t f'(h_t)-f'(h_t)x_{t+1}}=\frac{\beta\left[1+g'(x_{t+1})\right]}{x_{t+1}+g(x_{t+1})}. \tag{36}$$

Using (11) in (36) gives a relation between h_t and x_t, defined as $h_t = P(x_t)$. Hence h_{t+1} disappears from the Euler equation (12) so that our planar system (11)-(12) is reduced to a first-order nonlinear difference equation for x

$$x_{t+1} = x_t - P(x_t) + g(x_t). \tag{37}$$

Once the evolution of x is determined, the behavior of h can be obtained from (12) so that the system has become recursive. The slope of the first-order nonlinear difference equation (37) is

$$\frac{dx_{t+1}}{dx_t} = 1 - P'(x_t) + g'(x_t). \tag{38}$$

In the steady state $P(x) = g(x)$, so that equation (36) can be written as $f'(h)[x+g(x)] = \beta\left[1+g'(x)\right]\left[f(h)-f'(h)(x+h)\right]$. The steady state is not necessarily unique, since $g(x)$ is not monotone. We prove in Appendix 4 that the steady state Euler condition is an upward sloping curve in the hx-space and that the first-order nonlinear difference equation (37) is upward sloping, i.e. $1 - P'(x_t) + g'(x_t) > 0$. We summarize our findings in

Proposition 4. *Under the logarithmic utility function, the planar system reduces to a non-linear first-order difference equation for the natural resource stock. If the elasticity of output with respect to harvest is constant, the steady state equilibrium is unique. The equilibrium is stable regardless of whether the equilibrium is efficient or not.*

Proof: See Appendix 4.

The unique steady state equilibrium in our model with logarithmic preferences is stable. Since the initial condition for the resource stock is determined by history, this unique steady state and all the nonsteady-state equilibria tending towards it are determinate. Thus the qualitative properties of the equilibria with logarithmic preferences are very close to saddle point (and thus determinate) equilibria with quasi-linear utility.[12]

7. A Parametric Example

To shed further light on the properties of the our model with quasi-linear preferences, we use the following parametric example for the first period utility function, the production function and the resource growth function, respectively:

$$u(c_1) = \ln c_1 \tag{39}$$

$$f(h) = h^{\alpha}, \qquad 0 < \alpha < 1 \tag{40}$$

$$g(x) = ax - (1/2)bx^2 . \tag{41}$$

The economically interesting parameters are the output elasticity of resource (α), which determines the price elasticity of resource demand, and the discount factor (β). Equation (41) is the logistic growth function for renewable resources. With these specifications (13) and (14) reduce to

$$h = ax - (1/2)bx^2 \tag{42}$$

$$\frac{1}{(1-\alpha)h^{\alpha} - \alpha h^{\alpha-1}x} = \beta(1 + a - bx) . \tag{43}$$

The maximum growth, $\hat{x}$ equals a/b, and the respective harvest will be $(1/2)(a^2/b)$. We calculate the point, where the Euler equation hits the h-axis. Setting $x = 0$ we get

12 Indeterminacy often arises in models with stable and multiple equilibria. Indeterminacy in those models, however, is not caused by historically predetermined variables such as aggregate stocks of capital, human capital or resources, but by variables such as prices and interest rates, which are determined e.g. by expectations. [2, p. 450] has a short discussion about this distinction.

$h = \left[\left(\frac{1}{1-\alpha} \right) \left(\frac{1}{\beta(1+a)} \right) \right]^{\frac{1}{\alpha}} > 0$. Consider next the constellation of parameter values for which one of the steady states is the MSY. Plugging in the MSY values for the stock (a/b) and the harvest ($(1/2)(a^2/b)$) into equation (43) we obtain after a little manipulation the following relation between parameters

$$\ln \beta = -\alpha \ln \bar{h} - \ln \left[1 - \left(\frac{2+a}{a} \right) \alpha \right], \tag{44}$$

where $\bar{h} = (1/2)(a^2/b)$. Next we make specific assumptions about the values of parameters. Assumptions that $a = 1$ and $b = 0.001$ imply $\hat{x} = 1000$ and $\tilde{x} = 2000$. These values mean that the condition $1 + g'(x) \geq 0$ holds for all $0 \leq x \leq 2000$. Given these values, it follows that $\bar{h} = 500$. Then we get the relation between β and α depicted in Figure 5.

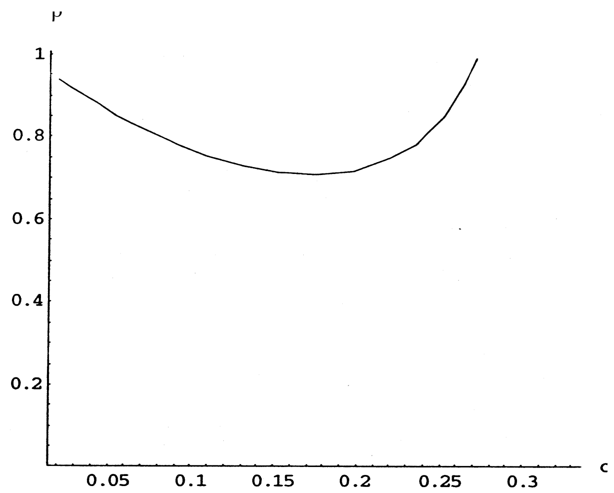

Figure 5. Relationship between β and α

Now we can pick up values of β and α from Figure 5 to get the MSY as one of the solutions. If $\alpha = 0.15$ and $\beta = 0.7158$ the other steady state stock is 1000, and the respective level of harvest 500. The second steady state in this case is the one where $x = 0.990433$ and $h = 0.989943$. Keeping the same value for the elasticity of output and decreasing the discount factor to $\beta = 0.70$, gives an example where the other steady state is efficient (the equilibrium with the lower level of the stock is in the parenthesis): $x = 985.374$ (1.15018) and $h = 499.893$ (1.14951). If we still keep the same value for the elasticity of output and let $\beta = 0.72$ we get an example where the other steady state is inefficient: $x = 1003.77$ $(.952382)$ and $h = 499.993$ (0.951929). Thus by using a parametric example we have demonstrated that under quasi-linear preferences with concave resource growth function there are two steady states, which may be efficient, inefficient or accidentally at the point of the maximum sustained yield.

8. CONCLUSIONS

We have examined an overlapping generations economy where a renewable natural resource stock serves as a store of value and an input in the production of consumption good. The resource grows according to the growth function, which we assume to be either concave or constant (when the resource is expendable). First we characterize the competitive equilibria of the economy under quasi-linear utility function and show that for a steady state equilibrium to exist, the discount factor may not be "too low", i.e., an economy with sufficient impatience extincts the resource. If the steady state exists the properties of equilibria depend crucially on the precise form of the resource growth function. For constant growth, there is at most one steady state, which is stable. For general concave growth there can be multiple steady states. If there are two steady states then the one associated with a larger stock is stable, and the other one associated with a smaller stock is unstable.

The unstable steady state is always efficient, but the stable one may or may not be. In particular, the steady state is inefficient if it lies to the right of the maximum sustained yield stock. Then the resource has been overaccumulated. In this case the growth rate of the resource is negative corresponding to the inefficiency results obtained in the overlapping generations models when the real interest rate is less than the population growth rate.

We also explored the robustness of our results with quasi-linear utility by assuming the periodic utility function to be logarithmic. In this case the dynamical system reduces to a non-linear first-order difference equation for the resource stock. In this case we show that for the Cobb-Douglas production function the steady state equilibrium is unique and stable regardless of whether the equilibrium is efficient or inefficient, and irrespective of the type of the growth function. Hence, the qualitative properties of the model under logarithmic utility function are similar to the saddle point equilibrium of the quasi-linear case.

Appendix 1: Proof of Lemmas 1 and 2

Lemma 1.

We rewrite the curve $f(h)-f'(h)(h+x)=0$ as $f\left[1-\frac{f'h}{f}\right]-f'x=0$. There are three possibilities for the limiting behavior. First, if $x\to 0$, then h must go towards some positive number. Second, if $h\to 0$, then x must approach some positive number. The last possibility is that the curve goes through the origin. In the first case, $\lim_{x\to 0} f\left[1-\frac{f'h}{f}\right]>0$ when h is a finite positive number. In the second case $\lim_{h\to 0} f\left[1-\frac{f'h}{f}\right]=0$, and u' is evaluated at $\lim_{h\to 0}(-f'(h)x)$, which is minus infinity. Thus equation must go through the origin. **Q.E.D**.

Lemma 2.

We let $x\to 0$. Then the resource growth function in the right-hand side of (18) in the text approaches some number. For the equation to hold the value of the left-hand side must then also approach some number. This happens for some finite h since the argument of the utility function can be rewritten as $f\left[1-\frac{f'h}{f}\right]$ (>0). Assume the contrary, i.e. that h is zero when $x\to 0$. From Lemma 1 we know that $\lim_{h\to 0} f\left[1-\frac{f'h}{f}\right]=0$. So when $x\to 0$ and $h\to 0$, the argument in the utility function approaches $\lim_{x\to 0, h\to 0}(-f'(h)x)$, which can be zero, minus infinity, or some negative number, so that the Euler equation cannot hold. **Q.E.D**.

* * * * * *

Appendix 2: Stability with Quasi-Linear Preferences

We analyze the stability of system (14) and (16).

$$x_{t+1}=G(x_t,h_t) \qquad \text{A.1}$$

$$h_{t+1}=F(x_t,h_t). \qquad \text{A.2}$$

The stability of the steady state depends on the eigenvalues of the Jacobian matrix of the partial derivatives

$$\mathrm{J} = \begin{bmatrix} G_x & G_h \\ F_x & F_h \end{bmatrix}.$$

Calculating the partial derivatives of the Jacobian matrix we get

$$G_x(x_t, h_t) = 1 + g'(x_t), \qquad G_h(x_t, h_t) = -1,$$

$$F_x(x_t, h_t) = [\beta\, f''(h_{t+1})]^{-1} \left[\frac{-(f'(h_t))^2 u''(c_t)[1+g'(x_t)]}{1+g'(x_{t+1})} - \frac{f'(h_t)u'(c_t)g''(x_{t+1})[1+g'(x_t)]}{[1+g'(x_{t+1})]^2} \right]$$

$$F_h(x_t, h_t) = [\beta\, f''(h_{t+1})]^{-1} \left[\frac{f''(h_t)u'(c_t)}{1+g'(x_{t+1})} + \frac{f'(h_t)u''(c_t)[f'(h_t) - f''(h_t)(x_t + g(x_t))]}{1+g'(x_{t+1})} \right]$$

$$+ [\beta\, f''(h_{t+1})]^{-1} \left[\frac{f'(h_t)u'(c_t)g''(x_{t+1})}{(1+g'(x_{t+1}))^2} \right].$$

Evaluating the elements of the Jacobian at the steady state and utilizing the facts that $u' = \beta(1+g')$ and $h = g$ we obtain

$$G_x(x,h) = 1 + g' \qquad G_h(x_t, h_t) = -1$$

$$F_x(x,h) = \frac{-(f')^2 u'' - f'\beta\, g''}{\beta\, f''} < 0$$

$$F_h(x,h) = 1 + \frac{(f')^2 u''}{(1+g')\beta\, f''} - \frac{f'u''(x+h)}{(1+g')\beta} + \frac{f'g''}{(1+g')\, f''} > 0$$

The determinant, D, and the trace of the Jacobian matrix, T, are D = $G_x F_h - G_h F_x$, and $T = G_x + F_h$, respectively. The characteristic polynomial is

$$p(\lambda) = \lambda^2 - (G_x + F_h)\lambda + (G_x F_h - G_h F_x) = 0, \qquad \text{A.3}$$

or expressed in terms of D and T

$$p(\lambda) = \lambda^2 - T\lambda + D = 0. \qquad \text{A.4}$$

From the stability theory of difference equations [2, pp. 63-67] we know that for a saddle point, the roots of $p(\lambda)=0$ need to be on both sides of unity. Thus we need that $D-T+1<0$ and $D+T+1>0$ or $D-T+1>0$ and $D+T+1<0$. The straightforward calculation yields at the steady state

$$D = 1 + g' - \frac{f'u''(x+h)}{\beta} > 0, \qquad \text{A.5}$$

$$T = 2 + g' + \frac{(f')^2 u''}{(1+g')\beta f''} - \frac{f'u''(x+h)}{(1+g')\beta} + \frac{f'g''}{(1+g')f''} > 1 \qquad \text{A.6}$$

meaning that $D+T+1>0$ holds. Calculating $D-T+1$ gives

$$D-T+1 = \frac{1}{\beta}(-f'u''(x+h)\left(\frac{g'}{1+g'}\right) - \frac{f'}{(1+g')f''}(f'u''+\beta g'')). \qquad \text{A.7}$$

To determine the sign of $D-T+1$ we compare the slopes of the growth curve and the consumer optimization condition at the steady state, calculated in equations (21) and (22) in the text (cf. Figures 2 and 3). At the larger steady state stock consumer first-order condition cuts the growth curve from below and we have

$$g' < \frac{u''f' + \beta g''}{-u''f''(h+x)}. \qquad \text{A.8}$$

This can be rearranged to yield

$$-f'u''f''(h+x)g' - f'(f'u''+\beta g'') > 0. \qquad \text{A.9}$$

Finally, dividing both sides by $f'' < 0$ and $1+g'$, we get

$$\frac{-f'u''(h+x)g'}{1+g'} - \frac{f'(f'u''+\beta g'')}{f''(1+g')} < 0, \qquad \text{A.10}$$

so that $D-T+1<0$, which is what is needed for a saddle point. Reversing A.8 leads to the condition $D-T+1>0$, where the steady state is a source (both eigenvalues exceed one).

* * * * *

Appendix 3: Efficiency

To solve the social planner's problem (PE) we form the following Lagrangean function

$$L = \phi c_2^0 + u(c_1) + \beta c_2 + \mu[f(h) - c_1 - c_2] + \theta[f(h) - c_1 - c_2^0] + \lambda_1[g(x) - h)] + \quad \text{B.1}$$

$$\lambda_2[x_1 + g(x_1) - x - h],$$

where μ, θ, λ_1, and λ_2 are nonnegative multipliers. Applying the Kuhn-Tucker theorem part of the first-order conditions are

$$u'(c_1) = \mu + \theta \qquad \text{B.2}$$

$$\beta = \mu \qquad \text{B.3}$$

$$\phi = \theta \qquad \text{B.4}$$

$$\mu f'(h) + \theta f'(h) - \lambda_1 - \lambda_2 = 0 \qquad \text{B.5}$$

$$\lambda_1 g'(x) = \lambda_2 . \qquad \text{B.6}$$

If the weight of the initial old generation, ϕ, is positive, we have $u'(c_1) = \mu + \theta$. Taking into account B.3, and the fact that $u'(c_1) = R\beta$ in steady state competitive equilibrium, the interest factor can be expressed as $R = 1 + \frac{\theta}{\beta} \geq 1$. Since $R = 1 + g'(x)$ holds in a steady state competitive equilibrium**,** efficient equilibria are characterized by the condition $R = 1 + g'(x) > 1$ i.e. $g'(x) \geq 0$, while those equilibria with $g'(x) < 0$ are inefficient.

* * * * *

Appendix 4: Stability and Uniqueness with Logarithmic Preferences

We first show that the difference equation (37) is upward sloping. Taking into account (11) we can rewrite (37) as

$$\begin{aligned} &f'(h)[x - h + g(x) + g(x - h + g(x))] = \\ &\beta[1 + g'(x - h + g(x)][f(h) - f'(h)x - f'(h)g'(x)]. \end{aligned} \qquad \text{C.1}$$

Denoting the future value of x by x' (i.e. $x' = x - h + g(x)$) we get

$$\{f''[x'+g(x')]-f'(1+g'(x'))+\beta g''(x')[f-f'(x+g(x))]-\beta(1+g'(x'))[f'-f''(x+g(x))]\}dh$$

$$=(1+g'(x))\{\beta g''(x')[f-f'(x+g(x))]-\beta(1+g'(x'))f'-f'(1+g'(x))\}dx \quad \text{C.2}$$

Denoting the term in braces on the left-hand side by Y (<0) and on the right-hand side by Z (<0) we obtain

$$\frac{dh}{dx}=P'(x)=\frac{[1+g'(x)]\{Z\}}{\{Y\}}>0\,. \qquad \text{C.3}$$

Note that $Y=Z+f''(x+g)[1+\beta(1+g')]$, which means that $0<P'(x)<1+g'$.

Next we evaluate C.3 in the steady state, where $x=x'$. To prove the stability of the steady state we need to have $1+g'(x)-P'(x)<1 \Leftrightarrow P'(x)>g'(x)$. This condition holds for all inefficient equilibria (where $g'(x)\leq 0$). As for the stability of efficient equilibria (where $g'(x)>0$), note first that if the steady state equilibrium is unique, then the upward sloping Euler equation must cut the resource growth curve from below so that the inequality (25) holds.[13] As will be shown below this is equivalent to the stability condition $P'(x)>g'(x)$.

First we characterize the Euler equation a bit more. In the logarithmic case the Euler curve goes through the point $\{h=0, x=0\}$.

Lemma A.1. The point $\{h=0, x=0\}$ fulfills the equation

$$f'(h)[x+g(x)]=\beta[1+g'(x)][f(h)-f'(h)(x+h)].$$

Proof. Suppose the Euler equation does not go through the origin. Then there are two possibilities for the limiting behavior. Either (i) if $x\to 0$, then h must go towards some positive number or (ii), if $h\to 0$, then x must approach some positive number. In the case (i) the right-hand side of C.1 approaches some number, because $\lim_{x\to 0} g'(x)$ is finite, but the left-hand side approaches zero and equation C.1 cannot hold. In the case (ii) when $h\to 0$ and $x\to 0$ the right-hand side approaches minus infinity, since $\lim_{h\to 0} f'(h)=\infty$, but the left-hand side approaches $\lim_{x\to 0, h\to 0}(f'(h)(x+h))$, which can be zero, infinity, or some positive number. **Q.E.D**.

- **Proof of stability:**

Total differentiation of the Euler condition yields

[13] Note that this is a necessary (but not sufficient) condition for uniqueness.

$$\frac{dh}{dx} = \frac{\{\beta g''[f - f'(x+h)] - (1+\beta)(1+g')f'\}}{f''(x+h)[1+\beta(1+g')]} > 0 . \qquad \text{C.4}$$

Note that this expression is not the same as the expression for $P'(x)$, since the latter expression was derived for points, which are valid outside the steady state, too.

For a unique steady state we need to have the slope of the Euler equation to cut the growth curve from below, i.e.

$$\frac{\{\beta g''[f - f'(x+h)] - (1+\beta)f'(1+g')\}}{f''(x+h)[1+\beta(1+g')]} > g' . \qquad \text{C.5}$$

Defining $f''(x+g)[1+\beta(1+g')] = a < 0$ makes it possible to re-express the slope of $P(x)$ as

$$P'(x) = \frac{[1+g'(x)]\{Z\}}{\{Z+a\}} , \qquad \text{C.6}$$

and the condition C.5 as

$$\frac{Z}{a} > g' , \qquad \text{C.7}$$

where $Z = \{\beta g''(x)[f - f'(x+g(x))] - (1+\beta)(1+g'(x))f'\} < 0$ at the steady state. Given C.5 we want to show that $P'(x) > g'$, i.e.

$$\frac{[1+g'(x)]\{Z\}}{\{Z+a\}} > g' . \qquad \text{C.8}$$

We have $\frac{Z}{a} > g' \Rightarrow \frac{a}{Z} < \frac{1}{g'}$. Adding unity to both sides gives $\frac{Z+a}{Z} < \frac{1+g'}{g'}$, so that $\frac{(1+g')Z}{Z+a} > g'$ **Q.E.D.**

- **Proof of uniqueness of the steady state with logarithmic preferences when the output elasticity is constant, i.e.** $f(h) = h^{\alpha}$ **:**

Rewriting equation (36) from the text at the steady state we get

$$\frac{f'(h)h + f'(h)x}{f(h) - hf'(h) - xf'(h)} = \beta[1+g'(x)]. \qquad \text{C.9}$$

Since $h = g(x)$ in the steady state, both sides of equation C.9 can be viewed as functions of x. The RHS is a decreasing function of x. We develop the LHS as follows

$$\frac{f'(h)h + f'(h)x}{f(h) - hf'(h) - xf'(h)} = \frac{\frac{x}{h}+1}{\frac{f}{f'h} - \frac{x}{h} - 1}, \qquad \text{C.10}$$

where the elasticity of output with respect to harvest is $f / f'h = 1/\alpha$. Using this we can rewrite C.10 as follows

$$LHS(x) = \frac{\frac{x}{g(x)}+1}{\frac{1}{\alpha} - \frac{x}{g(x)} - 1}. \qquad \text{C.11}$$

A straightforward differentiation yields

$$LHS'(x) = \frac{\frac{x}{g^2}(\frac{g}{x} - g')}{\frac{1}{\alpha} - \frac{x}{g} - 1} + \frac{(\frac{x}{g}+1)\left[\frac{x}{g^2}(\frac{g}{x} - g')\right]}{(\frac{1}{\alpha} - \frac{x}{g} - 1)^2}. \qquad \text{C.12}$$

Since the growth function is strictly concave, $LHS'(x) > 0$, and we have a unique equilibrium. Note that $f(h) - f'(h)h - f'(h)x$ is the first period consumption in a steady state, and thus it is positive, as is then the expression $f'h\left[\frac{f}{f'h} - \frac{x}{h} - 1\right]$ **Q.E.D.**

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