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Discussion papers

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CONSISTENT CROSS EXCHANGE RATES, NUMERAIRE PROBLEM, AND THE ROLE OF THE VEHICLE CURRENCY*

No. 76 5.1.1981

* Comments by Jouko Paunio and Seppo Honkapohja are gratefully acknowledged.

1. INTRODUCTION AND SUMMARY

This paper reconsiders the determination of the necessary and sufficient conditions for the establishment of consistent spot exchange rates across financial centers. Our starting point is Chacholiades's (1971, 1978) result according to which three-point arbitrage is all that is necessary to eliminate all arbitrage profits. According to Chacholiades this is the fundamental reason why K-point arbitrage ($K > 3$) is a rare phenomenon in the real world, if it takes place at all.

We shall show that this result does not hold in the general case when all exchange rates are quoted in all financial centers, and when each single exchange rate quotation is defined as a two-valued pair consisting of a bid-price as well as an ask-price. By allowing for the positive ask-bid spread the consistency criteria, *i.e.* the conditions by which profitable arbitrage opportunities are excluded, appear in the form of inequalities rather than in the form of equalities. This means that exchange rates need not be exactly the same in all centers at each moment of time, and still it may not be possible to make riskless profits by simultaneous sales and purchases of various currencies.

We identify foreign exchange dealers of commercial banks as the principal arbitragers. They are in the best position to be informed on quotations in different centers and therefore to seek for profit opportunities through arbitrage.[1] Their commercial customers are, of course, interested in buying the currency they need as cheaply as possible, but as far as dealers arbitrage efficiently, the customers have to choose from quotations which already are consistent. This helps to maintain the highly competitive nature of the foreign exchange market but as such is not necessary to ensure consistency.

When, because of the positive ask-bid spread associated with each exchange rate quotation, the consistency criteria are defined in terms of inequalities, it turns out that K-point arbitrage profit opportunities ($K > 3$) may arise even if all shorter arbitrage profit opportunities are excluded. This implies that, when there are many currencies the exchange rates between which are quoted in many centers, the amount of information needed for arbitragers to be able to check all possible arbitrage profit opportunities soon becomes enormous when the number of quotations grows. Furthermore, in the general case when all exchange rates are quoted in all centers, full information on quotations of a given center requires a matrix, each element of which includes both a bid-price and an ask-price. By these considerations it can safely be concluded that the computational difficulty faced by potential arbitragers is an important problem, which has to be solved in one way or another.

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1) Foreign exchange dealers' principal activity is to sell and buy currencies in retail transactions with their customers. As I have shown elsewhere (Suvanto, 1980), in this activity they frequently face situations when they need to buy a larger amount of some currency with some other currency (currencies) from other dealers. In these transactions they most probably buy the required currency from where it is cheapest. But these kind of wholesale transactions are different from pure arbitrage operations in which a given amount of one currency is, through simultaneous sales and purchases, ultimately exchanged for a larger amount of the <u>same</u> currency.

A natural way to economize the use of information on bilateral price ratios would seem to be to choose a unit of account by which the matrix of information can be reduced to a vector of information on prices expressed in terms of this *numeraire*. There can be essential saving in information costs only if dealers in all centers agree upon a common numeraire. In the present case, when each bilateral exchange rate quotation consists of two real numbers, the unit of account cannot be chosen arbitrarily. As a solution to the numeraire problem we suggest that the unit of account, or the reference currency, must be such that the ask-bid spread in the quotation of each currency *vis ā vis* the reference currency is equal to or smaller than the ask-bid spread in the quotation of this currency *vis ā vis* any other currency. But the ask-bid spread is a market determined price and tends to be the smaller the higher is the volume of trade with this currency. Therefore, the currency which can be chosen to serve as the reference currency must be the currency for which the market is the broadest. If such a reference currency can be found, then information on all cross exchange rates can be transmitted through a list of reference currency quotations. The consequence of this is that in any given center it is always as cheap to buy any currency j directly with currency i as to buy it indirectly via the reference currency. Frequent indirect transactions of this kind give the reference currency also the role of a vehicle currency.

When the numeraire problem has been solved according to the principles explained above, and when the role of the vehicle currency has been established, then it appears that, in fact, two-point arbitrage via the vehicle currency is sufficient to ensure consistent cross exchange rates between financial centers. This does not imply that opportunities for larger profits through longer chains would never arise, but these opportunities are unlikely to be realized, because two-point profit opportunities are easier to be observed, and once observed and realized exchange rates will move so that opportunities for even larger profits will disappear.

2. CRITERIA FOR CONSISTENCY

We shall adopt the following notation:

$s^a_{ij}(m)$ ask-price of currency j in terms of currency i as quoted in center m (dimension i/j);

$s^b_{ij}(m)$ bid-price of currency j in terms of currency i as quoted in center m (dimension i/j);

$\rho_{ij}(m)$ ask-bid spread in proportion to the ask-price, i.e. $\rho_{ij}(m) = [s^a_{ij}(m) - s^b_{ij}(m)]/s^a_{ij}(m)$.

A single quotation is denoted by a pair $\{s^b_{ij}(m),\ s^a_{ij}(m)\}$. All ask and bid prices are positive real numbers.

There are N+1 currencies, $i,j = 0,1,2,\dots,N$, and M financial centers, $m = 1,2,\dots,M$.

We set the following eight criteria for the consistency of the cross exchange rates within and between financial centers:

C1 $$s^a_{ij}(m) > s^b_{ij}(m) \Leftrightarrow 0 < \rho_{ij}(m) < 1 \qquad (i \neq j)$$

C2 $$s^a_{ii}(m) = s^b_{ii}(m) = 1$$

C3 $$s^a_{ij}(m)\,s^b_{ji}(m) = 1 \Leftrightarrow s^b_{ji}(m) = 1/s^a_{ij}(m)$$

C4 $$\begin{cases} s^a_{ij}(m)\,s^a_{ji}(m) > 1 \\ s^b_{ij}(m)\,s^b_{ji}(m) < 1 \end{cases}$$

C5 $$\begin{cases} s^a_{ij}(m) \leq s^a_{ik}(m)\,s^a_{kj}(m) \\ s^b_{ji}(m) \geq s^b_{jk}(m)\,s^b_{ki}(m) \end{cases}$$

C6 $$\begin{cases} s^a_{ij}(m)\,s^a_{ji}(n) \geq 1 \\ s^b_{ij}(n)\,s^b_{ji}(m) \leq 1 \end{cases}$$

C7 $$\begin{cases} s^a_{ij}(m)\,s^a_{jk}(n)\,s^a_{ki}(p) \geq 1 \\ s^b_{ik}(p)\,s^b_{kj}(n)\,s^b_{ji}(m) \leq 1 \end{cases}$$

C8 $$\begin{cases} s^a_{ih}(m)\,\underbrace{s^a_{hk}(n)\,s^a_{kl}(p)\ \ldots}_{\text{K ask-prices } (K > 3)}\ s^a_{ji}(q) \geq 1 \\ s^b_{ij}(q)\ \underbrace{\ldots\ s^b_{lk}(p)\,s^b_{kh}(n)}_{\text{K bid-prices } (K > 3)}\,s^b_{hi}(m) \leq 1\,. \end{cases}$$

The first criterion C1 tells that a positive ask-bid spread is associated with each single quotation. The interpretation of the second criterion C2 is trivial. The third criterion C3 tells that in any given center the ask-price of j in terms of i is equal to the inverse of the bid-price of i in terms of j. The fourth criterion C4 follows directly from C1 and C3.

The fifth criterion C5 tells that in any given center it is always at least as cheap to buy j directly with i than to buy it indirectly via some third currency.

The last three criteria concern the consistency of exchange rates between different financial centers. The sixth criterion C6 excludes the possibility of making profits by *two-point arbitrage*, i.e. by buying j with i in center m and simultaneously buying i with j in some other center n. This means that the ask-price of j in terms of i has to be everywhere higher than the corresponding bid-price in any of the centers, or

$$s^a_{ij}(m)\,s^a_{ji}(n) = s^a_{ij}(m)/s^b_{ij}(n) \geq 1$$

$$\Rightarrow\; s^a_{ij}(m) \geq s^b_{ij}(n) \quad \text{(for all m and n)}.$$

The seventh criterion C7 excludes the profit opportunities by *three-point arbitrage*, and the last criterion C8 generalizes this into the case of K currencies and K centers and excludes *K-point arbitrage* profits.

Note that the arbitrage chain which consists of only ask-prices expresses the price (in terms of i) the arbitrager has to pay if he buys i through simultaneous sales and purchases of other currencies starting with currency i. Similarly, the arbitrage chain which consists of only bid-prices expresses the amount of i the arbitrager receives through simultaneous sales and purchases of other currencies when he starts with first selling a unit of currency i.[2] Note also that the terms in criteria C6 - C8 can be rearranged, which implies that if there is an opportunity for an arbitrage profit, it does not matter which is the first currency in the chain. As a matter of fact, there is no chain in the real time, because sales and purchases are made simultaneously.

According to these criteria exchange rates can be consistent even though they need not be exactly the same in all financial centers. Because exchange rates are prices which are set by foreign exchange dealers every time when quotations are asked, they may change, and therefore profit opportunities may temporarily arise. We assume that dealers behave so that once any of them observes a profit opportunity, he will immediately realize it, which action itself gives a signal for other dealers to change quotations so that the profit opportunity will immediately disappear. Because each dealer has only limited resources invested in various currencies, none of them can realize but a finite profit, which quarantees the stability of the system.

2) A numerical example is given in the Appendix (Example 1). The quotations are chosen arbitrarily. However, the ask and bid prices as well as the ask-bid spreads correspond closely to those prevailing in the mid December 1980.

3. CHACHOLIADES'S THEOREM REVISITED

Chacholiades (1971, 1978) paid attention to the inherent complexity of observing K-point arbitrage opportunities when K is large, but he did not accept the view that computational difficulty as such would be a deterrent to arbitragers for attempting easy profits in an age where the use of computers is widely spread. Instead he sought for a more fundamental reason why K-point arbitrage is a rare phenomenon in the real world. For this he proved a theorem according to which three-point arbitrage is all that is needed to eliminate all profitable arbitrage opportunities. In the following we shall show that his does not hold in the general case when the allowance is made for the positive aks-bid spread. Indeed, it will turn out that all kind of arbitrage profit opportunities may arise independently. To show this we prove a series of simple propositions.

<u>P1</u> Three-point arbitrage may be profitable even if no two-point arbitrage is profitable.

<u>Proof</u>. By assumption it holds that

$$[\underbrace{s^a_{ij}(m)s^a_{ji}(n)}_{\geq 1}][\underbrace{s^a_{jh}(p)s^a_{hj}(q)}_{\geq 1}][\underbrace{s^a_{hi}(r)s^a_{ih}(s)}_{\geq 1}] \geq 1 \qquad (1)$$

Rearranging the terms this can be written

$$[s^a_{ij}(m)s^a_{jh}(p)s^a_{hi}(r)][s^a_{ji}(n)s^a_{ih}(s)s^a_{hj}(q)] \geq 1 \qquad (2)$$

One, but only one, of these three-point chains may be < 1, which completes the proof.

<u>P2</u> Two-point arbitrage can be profitable even if no three-point arbitrage is profitable.

<u>Proof.</u> This can be proven be reversing the argument of the previous proof. First assume that the two three-point chains in equation (2) are both ≥ 1 and then rearrange the terms to arrive at equation (1). One or two, but not all, of the resulting two-point chains may be < 1, which completes the proof.

<u>P3</u> Four-point arbitrage can be profitable even if neither two-point nor three-point arbitrage is profitable.

<u>Proof.</u> By assumption it holds that

$$\underbrace{[s^a_{ij}(m)\, s^a_{jh}(n)\, s^a_{hi}(p)]}_{\geq 1}\underbrace{[s^a_{hk}(q)\, s^a_{ki}(r)\, s^a_{ih}(s)]}_{\geq 1} \geq 1. \tag{3}$$

Rearranging the terms we get

$$\underbrace{[s^a_{ij}(m)\, s^a_{jh}(n)\, s^a_{hk}(q)\, s^a_{ki}(r)]}_{\gtrless 1}\underbrace{[s^a_{hi}(p)\, s^a_{ih}(s)]}_{\geq 1} \geq 1. \tag{4}$$

The two-point chain in the latter brackets is ≥ 1 by assumption, but the four-point chain in the former brackets may be < 1, which completes the proof.[3]

The last result can be easily generalized to all K-point abritrage chains $(K > 3)$.

P4 K-point arbitrage $(K > 3)$ may be profitable even if no J-point arbitrage $(3 \leq J < K)$ is profitable.

Proof. Because (K-1)-point arbitrage is not profitable then it holds that

$$\underbrace{s^a_{ij}(1)s^a_{jh}(2) \dots s^a_{st}(K-3)s^a_{tu}(K-2)s^a_{ui}(K-1)}_{\text{K-1 ask-prices}} \geq 1. \tag{4}$$

It also holds that

$$s^a_{uv}(K)s^a_{vi}(K+1)s^a_{iu}(K+2) \geq 1. \tag{5}$$

Multiplying these two expressions together and rearranging the terms we get

$$[\underbrace{s^a_{ij}(1)s^a_{jh}(2) \dots s^a_{st}(K-3)s^a_{tu}(K-2)s^a_{uv}(K)s^a_{vi}(K+1)}_{\text{K ask-prices}}]$$

$$\times\ [s^a_{ui}(K-1)s^a_{iu}(K+2)] \geq 1. \tag{6}$$

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3) A numerical example of this case is shown in the Appendix (Example 2).

The two-point chain in the latter brackets is ≥ 1 by assumption, but the K-point chain in the former brackets may be < 1, which completes the proof. Note that P3 is a special case of this result, whereas P1 is not.

P5 (K-1)-point arbitrage may be profitable even if neither two-point, three-point nor K-point arbitrage ($K > 4$) is profitable.

Proof. This can be proven by reversing the argument of the previous proof.

We have now shown that short arbitrage chains, i.e. arbitrage with two or three currencies only, does not necessarily quarantee that all arbitrage profit opportunities would be eliminated.[4] Of course, even two-point arbitrage would keep exchange rate quotations in different centers very close to each others, so that for most practical purposes they could be regarded as equal. But, if it is true, as Chacholiades claims, that computational difficulty associated with observing longer arbitrage profit opportunities should not be a deterrent for capturing easy profits, then we

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4) The reason why Chacholiades got a different result was that he did not allow for the positive ask-bid spread and that he did not index exchange rates according to the centers in which they are quoted and therefore assumed, without discussion, that opportunities for two-point arbitrage profits will never arise. As a consequence, the inequality criteria C4 - C8 should be replaced by equality criteria. If '≥ 1' in equations (4) and (5) were replaced by '$= 1$', the expression (6) would also be $= 1$. Then because two-point chain in the latter brackets would by assumption be $= 1$, the K-point chain in the former brackets should also be $= 1$, implying that K-point arbitrage would not be profitable if (K-1)-point arbitrage profits were excluded.

should assume that dealers in fact seek for these opportunities and utilize them whenever they occur. But the computational difficulty faced by a potential arbitrager actively seeking for profit opportunities via longer arbitrage chains is a huge problem even with the help of computers, and therefore it may well be that some opportunities remain unutilized implying that exchange rates are not strictly consistent.

To demonstrate the nature of the information problem let us assume that a potential arbitrager wants to check all two-point, three-point and four-point arbitrage profit opportunities when all N+1 currencies are quoted in all M centers. To check all two-point opportunities the arbitrager has first to choose $s^a_{ij}(m)$, which can be done in M(N+1)N different ways, and then to choose $s^a_{ji}(n)$ from any of the remaining M-1 centers ($n \neq m$). The number of possible combinations is (N+1)NM(M-1), but because $s^a_{ij}(m)s^a_{ji}(n) = s^a_{ji}(n)s^a_{ij}(m)$, only half of them are independent. To check all three-point chains the arbitrager has first to choose $s^a_{ij}(m)$ as above from M(N+1)N alternatives, and then to choose the second quotation $s^a_{jh}(m)$ from (N-1)(M-1) alternatives ($h \neq i$; $n \neq m$), and finally to choose $s^a_{hi}(p)$ from any of the remaining M-2 centers ($p \neq n,m$).[5] Because any three-point chain can be written in three different orders, the number of independent combinations

5) The restriction that all quotations should be taken from different centers is not strictly necessary. This would increase the number of possible combinations even further.

is $\frac{1}{3}[(N+1)N(N-1)M(M-1)(M-2)]$. To check all four point chains the arbitrager has to choose the first and second quotation, $s^a_{ij}(m)$ and $s^a_{jh}(n)$, as above, and then to choose the third quotation $s^a_{hk}(p)$ from (N-2)(M-2) alternatives ($k \neq i,j,h$: $p \neq m,n$), and finally to choose the last quotation $s^a_{ki}(q)$ from any of the remaining M-3 centers ($q \neq m,n,p$). Because any four-point chain can be rearranged in four different ways, the number of independent combinations is $\frac{1}{4}[(N+1)N(N-1)(N-2)M(M-1)(M-2)(M-3)]$.

As shown below the number of possible independent combinations increases extremely rapidly as the number of currencies and the number of centers increase.

Number of centers M	Number of currencies N+1	Number of combinations Two-point	Three-point	Four-point
10	20	17 100	1 641 600	146 512 800
10	10	4 050	172 800	6 350 400
5	10	900	14 400	151 200
5	5	200	1 200	3 600

4. NUMERAIRE PROBLEM AND THE CHOICE OF THE VEHICLE CURRENCY

Above we have implicitly assumed that foreign exchange dealers receive quotations from each center in the form of a matrix, the element of which is a single quotation $\{s^b_{ij}(m),\ s^a_{ij}(m)\}$. This is apparently a very uneconomic way of transmitting information. We also demonstrated that to check all potential profit opportunities, even though short arbitrage chains only, involves a great computational diffuculty. Therefore, the market participants have a good case for economizing the use of information.

A normal way to economize information on the matrix of bilateral prices (terms of trade) is to choose one of the goods, in the present case one of the currencies, for a *numeraire*, i.e. a unit of account in terms of which the prices of all other goods are expressed.

Let us try to apply this procedure to the quotation matrix. We assume that dealers have agreed that all of them quote all N currencies, $i = 1,2,\ldots,N$, *vis á vis* an arbitrarily chosen common reference currency o. Thus dealers in center p give a vector of reference currency quotations:

$$\{s^b_{io}(p),\ s^a_{io}(p)\} \qquad (i = 1,\ldots,N)$$

These are assumed to satisfy the criteria C1 - C4. What then are the cross exchange rates in center p consistent with this vector of quotations? For example, the price of k in terms of j can be chosen in four different ways:

(a) $$s^b_{jo}(p)/s^b_{ko}(p) = s^b_{jo}(p)s^a_{ok}(p)$$

(b) $$s^b_{jo}(p)/s^a_{ko}(p) = s^b_{jo}(p)s^b_{ok}(p)$$

(c) $$s^a_{jo}(p)/s^b_{ko}(p) = s^a_{jo}(p)s^a_{ok}(p)$$

(d) $$s^a_{jo}(p)/s^a_{ko}(p) = s^a_{jo}(p)s^b_{ok}(p)\ .$$

Because by criterion C1 $s^b_{jo}(p) < s^a_{jo}(p)$ and $s^b_{ok}(p) < s^a_{ok}(p)$, it is seen that from the four alternatives (b) is the smallest and (c) is the biggest.

Assume that a dealer in center p receives a sell order from a customer who wants to buy a given amount of k with j and that the reference currency quotations are already given. The dealer's interest in this situation is to ask for such a price of k that he will receive as much of j as possible.[6] Therefore, the ask-price of k in terms of j is

$$s^a_{jk}(p) = s^a_{jo}(p)s^a_{ok}(p) . \tag{7}$$

If he asked for a higher price than this the customer would refuse to buy because he would have an option first to buy o with j and then to buy k with o. Assume then that the dealer receives a buy order from a customer who wants to sell a given amount of k against j. Now the dealer's interest is to bid for such a price of k that he has to give away as little of j as possible. Therefore, the bid-price of k in terms of j is

$$s^b_{jk}(p) = s^b_{jo}(p)s^b_{ok}(p) . \tag{8}$$

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6) This way of determining cross exchange rates from a list of reference currency quotations is illustrated by practical examples by Hudson (1979, p. 40).

If he would bid for a lower price than this the customer would refuse to sell because he would have an option first to sell k against o and then to sell o against j.

The proportionate ask-bid spread $\rho_{jk}(p)$ is consequently

$$\rho_{jk}(p) = 1 - \frac{s^b_{jo}(p)\, s^b_{ok}(p)}{s^a_{jo}(p)\, s^a_{ok}(p)} \tag{9}$$

$$= 1 - \frac{s^a_{jo}(p)(1 - \rho_{jo}(p))\;\; s^a_{ok}(p)(1 - \rho_{ok}(p))}{s^a_{jo}(p)\, s^a_{ok}(p)}$$

$$= 1 - (1 - \rho_{jo}(p))(1 - \rho_{ok}(p))$$

$$= \rho_{jo}(p) + \rho_{ok}(p) - \rho_{jo}(p)\rho_{ok}(p)$$

Because the proportionate ask-bid spread is a very small number (of order 0.01 or smaller), the cross term $\rho_{jo}(p)\rho_{ok}(p) \cong 0$. It is easy to show that the proportionate ask-bid spread does not depend on the dimension, or $\rho_{ij}(p) = \rho_{ji}(p)$. Taking these two aspects into account equation (9) can be written as

$$\rho_{jk}(p) \cong \rho_{jo}(p) + \rho_{ko}(p)\,. \tag{10}$$

Because the ask-bid spread has to be positive it follows from equation (10) that $\rho_{jk}(p)$ must be bigger than either $\rho_{jo}(p)$

or $\rho_{ko}(p)$. This must hold for all j and k or

$$\begin{cases} \rho_{jo}(p) = \min\limits_{k} \rho_{jk}(p) \\ \rho_{ko}(p) = \min\limits_{j} \rho_{kj}(p) \, . \end{cases} \tag{11}$$

The ask-bid spread can be regarded as a market determined price for dealers' services, and in the competitive market it depends on the volume of trade between currencies so that it is the smaller the higher is the volume of trade.[7] Recall that so far the reference currency o has been arbitrarily chosen. Equation (11) shows that this cannot be done. Assume that there are two currencies 1 and 2, between which the market is so broad that the competitive ask-bid spread $\hat{\rho}_{12}(m)$ becomes smaller than the ask-bid spread that can be derived from the reference currency quotations for these currencies, or $\hat{\rho}_{12}(m) < \rho_{10}(m) + \rho_{20}(m)$ is true at least in one center m. This would imply that 1 and 2 would be quoted *vis á vis* each other independently of the reference currency quotations, which in turn would imply some loss of the saving in information costs achieved through the use of a common numeraire. The more these kinds of cases can be found the more questionable the reference currency becomes.

We conclude that a necessary condition for the establishment of a common numeraire is that it is possible to find such a currency for which the conditions expressed by equation (11) are in force and are backed by market forces. When this kind

7) This can be rationalized by economies of scale in foreign exchange dealing, cf. Suvanto (1980).

of reference currency can be found and when cross exchange rates are determined in accordance to equations (7) and (8), then anyone who wants to buy k with j in center p can do this either directly or indirectly via the reference currency. Both ways are equally cheap. On the other hand, according to the criterion C5 these two ways are always at least as attractive as to buy k with j via some third currency $h \neq o$.

The implication of the above reasoning is that when it is possible to agree upon a common reference currency, then this currency will be used also as a *vehicle currency*, which means that currencies will be bought and sold via this currency. This in turn makes the market for this currency broader than it would be otherwise thus supporting the smaller ask-bid spread which is necessary for maintaining this role. The bredth of the market is not completely independent of the size of the domestic money market for this currency. Because transactions in the vehicle currency are reflected in the demand deposits with the commercial banks of the country where the vehicle currency is used for domestic transactions, large international transactions with this currency must be small enough not to disturb domestic money markets considerably.[8] Furthermore, the international transactors must have a strong expectation that the authorities of the home

8) McKinnon (1979, p. 34-35) has rationalized the role of the US dollar as an intervention currency used by central banks by the similar kind of argument.

country of the vehicle currency will not restrict the non-resident convertibility of that currency (cf. Dufey and Giddy, 1978, p. 262). By these considerations the role of the US dollar as a vehicle currency becomes understandable.

Now, if we define, as we did above, arbitrage as a chain of simultaneous sales and purchases of currencies starting from one currency and returning to this same currency, and if information on exchange rates is transmitted as a vector of reference currency quotations, and if cross exchange rates can be determined consistently on the basis of this information, then we come to the question of whether this information alone is sufficient to reveal arbitrage profit opportunities when they occur. Next we shall show that this indeed is the case and that two-point arbitrage alone is sufficient to keep exchange rates consistent.

> P4 If the bredth of the market allows for the choice of a common reference currency, and if cross exchange rates are determined on the basis of the reference currency quotations according to equations (7) and (8), then, if there is any arbitrage profit opportunity, there is at least one two-point arbitrage profit opportunity between some currency i and the reference currency.

Assume first that there is a profit opportunity through two-point arbitrage between currencies i and j in centers m and n, or

$$s^a_{ij}(m)s^a_{ji}(n) < 1 \qquad (i, j \neq o) .$$

Applying equation (7) this can be rewritten as follows

$$s^a_{io}(m)s^a_{oj}(m)s^a_{jo}(n)s^a_{oi}(n) < 1$$

or rearranging the terms

$$[s^a_{io}(m)s^a_{oi}(n)][s^a_{jo}(n)s^a_{oj}(n)] < 1 .$$

Because the whole expression is < 1, at least one of the expressions within the brackets must be < 1. This completes the proof on the part of two-point arbitrage. Assume next that there are three currencies i, j and k the quotations between which in centers m, n and p allow an opportunity for three-point arbitrage profit, or

$$s^a_{ij}(m)s^a_{jk}(n)s^a_{ki}(p) < 1 \qquad (i , j \neq o).$$

Applying again equation (7) this can be rewritten as follows

$$s^a_{io}(m)s^a_{oj}(m)s^a_{jo}(n)s^a_{ok}(n)s^a_{ko}(p)s^a_{oi}(p) < 1$$

and rearranging the terms

$$[s^a_{io}(m)s^a_{oi}(p)][s^a_{jo}(n)s^a_{oj}(m)][s^a_{ko}(p)s^a_{ok}(n)] < 1 .$$

Because the whole expression is < 1, at least one of the two-point chains within the brackets must be < 1, which completes the proof on the part of three-point abritrage. In the similar way any given K-point arbitrage chain which is < 1 can be decomposed into two-point chains via the reference currency, at least one of which must be < 1.

According to this result any longer arbitrage chain (K > 2) is redundant to ensure consistent exchange rates between centers in the case where information is transmitted through the reference currency quotations only. This result does not exclude possibilities for even larger profits through three point arbitrage. For example, if the first two-point chains in the last expression both are < 1, then a three-point chain, say, $s^a_{jo}(n)s^a_{oi}(p)s^a_{ij}(m)$ would give a larger profit than either $s^a_{jo}(n)s^a_{oj}(m)$ or $s^a_{io}(m)s^a_{oi}(p)$ alone. But because dealers receive direct information on the reference currency quotations only, it seems natural to assume that two-point profit opportunities are already observed before anybody has had time to check eventual longer profit opportunities.

Computational difficulty in this case is, in fact, rather small and well manageable even keeping in mind that decisions to arbitrage have to be made very quickly. To check all potential profit opportunities in the case of N+1 currencies and M centers one has first to choose the first quotation $s^a_{io}(m)$ from MN alternatives ($i \neq 0$) and then to choose the second quotation $s^{a}_{oi}(p)$ from any of the remaining M-1 centers ($p \neq m$). The number of possible combinations is hence MN(M-1). The table below demonstrates the dramatic reduction of the dimension of the information problem as compared with the general case.

Number of centers M	Number of currencies N+1	Number of combinations MN(M-1)
10	20	1 710
10	10	810
5	10	180
3	5	24
2	3	8

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APPENDIX NUMERICAL EXAMPLES

EXAMPLE 1.

Center	Bid-price	Ask-price	Dimension	Proportionate ask-bid spread
m	$s^b_{ij}(m) = 0.4287$	$s^a_{ij}(m) = 0.4292$	GBP/USD	0.0012
n	$s^b_{ij}(n) = 0.5027$	$s^a_{ij}(n) = 0.5032$	USD/DEM	0.0010
p	$s^b_{ij}(p) = 4.6225$	$s^a_{ij}(p) = 4.6325$	DEM/GBP	0.0022

$s^a_{ij}(m)s^a_{jh}(n)s^a_{hi}(p) = 1.0005 > 1 \Rightarrow$ Buying GBP 1 mill. via USD and DEM costs GBP 1000500.

$s^b_{ij}(m)s^b_{jh}(n)s^b_{hi}(p) = 0.9962 < 1 \Rightarrow$ Selling GBP 1 mill. via USD and DEM brings GBP 996200.

Therefore, three-point arbitrage is not profitable given the above quotations.

EXAMPLE 2.

Center	Ask-price	Dimension	Center	Ask-price	Dimension
m	$s^a_{ij}(m) = 0.4287$	GBP/USD	r	$s^a_{hk}(r) = 1.0975$	DEM/CHF
n	$s^a_{jh}(n) = 0.5032$	USD/DEM	s	$s^a_{ki}(s) = 4.2100$	CHF/GBP
p	$s^a_{hi}(p) = 4.6325$	DEM/GBP	t	$s^a_{ih}(t) = 0.2165$	GBP/DEM

$$\underbrace{[s^a_{ij}(m)s^a_{jh}(n)s^a_{hi}(p)]}_{=\,1.0005}\underbrace{[s^a_{hk}(r)s^a_{ki}(s)s^a_{ih}(t)]}_{=\,1.0003} = 1.0008 > 1$$

$$\underbrace{[s^a_{ij}(m)s^a_{jh}(n)s^a_{hk}(r)s^a_{ki}(s)]}_{=\,0.9979}\underbrace{[s^a_{hi}(p)s^a_{ih}(t)]}_{=\,1.0029} = 1.0008 > 1$$

Four-point arbitrage with GBP 1 mill. will bring a profit of GBP 2100.

There is no opportunity for two-point or three-point arbitrage profits given the above quotations.