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1. Introduction

Methods of measuring the real growth of industrial production or gross domestic product are various. Although it is usually admitted that some kind of double deflation method of decomposing the value added produced e.g. in paper industry into price and quantity components is theoretically most plausible, in practice more straightforward methods are often applied. Hill (1971, p. 9) writes that "the SNA has never offered any explicit or spesific guidance on what the constant price data are supposed to measure, and countries are quite free to adopt whatever conventions they please".

As an example we consider the industrial production of paper industry. Its gross production (or gross output) consists of various paper products sold for other industries (or stored). The value of gross output (in current prices) is usually measured at producers' prices. When the value of raw materials, energy and other intermediate material inputs used in the production of gross output are substracted from its value, we get the value added in paper industry. The value added is the result of labour and capital services and its value in current prices divides essentially into compensations of these

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services. Therefore the value added in current prices is defined as the value of the output in the industry less the value of the intermediate material inputs consumed and it is used up for labour and capital compensations needed in its production. In other words: value of gross output = value of the intermediate material inputs + value of the value added (or value of the labour and capital services[1]).

The double deflation method of calculating the volume of value added, or the real product of an industry, uses its value added measured at constant prices. "Value added at constant prices can be derived as the difference between gross output and total intermediate inputs, each mesured at constant prices, in exactly the same way that value added is derived at current prices", as Hill (1971, p. 13) has put it. The expression "double deflation" originates from the idea that both the value of gross output and the value of intermediate inputs are deflated by appropriate price indices to express these in constant prices. However, as Hill also stresses, this is an unfortunate choice of words, because there is no necessity to use price indices and "double" (i.e. two) deflations. Gross output and total intermediate inputs can be measured straighty in constant prices e.g. by multiplying their volume indices (perhaps calculated without using price indices) by their base year values.

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1) More accurately: Compensation of employees + consumption of fixed capital + indirect taxes - subsidies + operating surplus (a residual).

Note also that decomposition of the value added to its components does not help us here, Hill (1971, p. 14):

> Although value added at current prices can be expressed as the sum of four components, compensation of employees, operating surplus, consumption of capital and net indirect taxes, there is no way in which each of these four components can be separately measured at constant wage or tax rates in such a way as to preserve the identity with value added. Indeed, there is no conceivable way in which the operating surplus, being a pure residual, can be expressed at its own constant prices as there is no quantity unit and hence no price per unit.

Instead of double deflation method it is often assumed (usually without a detailed analysis), that the real value added changes in accordance with either the real gross output or the volume of intermediate inputs. These "single indicator methods", see Hill (1971, p. 14), propose the use of volume index of gross output or of intermediate inputs as a proxy for the volume index of value added. These methods should be considered only as substitutes the use of which may be accepted only if the data basis does not allow the application of more appropriate double deflation methods.

However, the double deflation method described above is by no means the only possibility. It is a straightforward generalization of the constant price technique of national accounts. The constant price series usually express a variable (say private consumption) in prices of some base year, say 1964. If $\Sigma\, p_i^t q_i^t = p^t \cdot q^t$ is the value of private consumption in current prices p^t, then the value of private consumption in 1964 prices is $\Sigma\, p_i^{64} q_i^t = p^{64} \cdot q^t$. This, of course, equals the Laspeyres' quantity index $Q_{64}^t(La) = \Sigma\, p_i^{64} q^t / \Sigma\, p_i^{64} q_i^{64} = p^{64} \cdot q^t / p^{64} \cdot q^{64}$

of private consumption multiplied by consumption expenditures $C^{64} = p^{64}\cdot q^{64}$ in 1964: $p^{64}\cdot q^{t} = C^{64}Q_{64}^{t}(La)$. It may be calculated also by deflating the consumption expenditures $C^{t} = p^{t}\cdot q^{t}$ by Paasche's price index $P_{64}^{t}(Pa) = \Sigma\, p_i^t q_i^t / \Sigma\, p_i^{64} q_i^t = p^t\cdot q^t/p^{64}\cdot q^t$, i.e. $p^{64}\cdot q^t = C^t/P_{64}^t(Pa)$. Therefore the constant price series of national accounts lean heavily on either the Laspeyres' volume index or the Paasche's price index. These indices may, however, give rather biased results. E.g. the Laspeyres' quantity index

$$\text{(1)} \qquad Q_0^t(La) = p^0\cdot q^t/p^0\cdot q^0$$

$$= \Sigma\, p_i^0 q_i^t/\Sigma\, p_j^0 q_j^0$$

$$= \Sigma\, \frac{p_i^0 q_i^0}{\Sigma\, p_j^0 q_j^0}\,(q_i^t/q_i^0)$$

$$= \Sigma\, w_i^0 (q_i^t/q_i^0)$$

gives certainly correct results only if either all quantities change in the some proportion (i.e. $q_i^t = \kappa q_i^0$) or prices have changed proportionally (i.e. $p_i^t = \lambda p_i^0$). Its unbiasedness has very little to do with the constancy of value shares (i.e. $w_i^t = p_i^t q_i^t/p^t\cdot q^t = w_i^0$), although this is a generally occuring missconception. People tend to think that Laspeyres' index (whether for prices or quantities) remains unbiased as long as the new and old value shares remain approximately equal, $w_i^t \approx w_i^0$. This is usually false, because in the case of constant value shares the weighted geometric means

$$(2) \qquad Q_0^t = \Pi(q_i^t/q_i^0)^{w_i^0} = \exp(\Sigma\ w_i^0 \log\ (q_i^t/q_i^0))$$

$$(3) \qquad Q_0^t = \Pi(q_i^t/q_i^0)^{w_i^t} = \exp(\Sigma\ w_i^t \log\ (q_i^t/q_i^0))$$

are both unbiased (and approximately equal), not the arithmetic mean $Q_0^t(La) = \Sigma\ w_i^0(q_i^t/q_i^0)$ exceeding the geometric mean Q_0^t of (2) (unless $q_i^t/q_i^0 \approx$ constant for all i). The somewhat too positive additude towards Laspeyres' and Paasche's indices stems largely from Fisher's pioneering work The Making of Index Numbers published in 1922. In Fisher's calculations Laspeyres' and Paasche's indices did not reveal their biases which made Fisher to regard them incorrectly as almost unbiased, see Vartia (1978).

It seems now appropriate to consider what kind of biases may result in the volume series of national accounts and especially in real value added from the use of these indices. Our aim is to develop a more symmetrical, consistent and unbiased double deflation method than the one based on Laspeyres' quantity index. Of course, the increased accuracy and symmetry costs something in terms of simplicity in calculations. But it seems to be the time to start to improve the constant price methods of national accounts, which have now served us a few decades, by more accurate and sophisticated index number calculations.

2. Derivation of the new double deflation method

Let's consider first any positive variables $\bar{V}$ and X measured in ratio scale and their difference

$$(4) \qquad V = \bar{V} - X ,$$

which is also assumed to be positive, $V > 0$, i.e., $\bar{V} > X$. In terms of the value added problem the $\bar{V}$, X and V are values of the gross output, intermediate material input and value added, respectively. But for a while they may be any similar variables. Arithmetical changes in the variables satisfy

$$(5) \qquad \Delta V = \Delta\bar{V} - \Delta X$$

or more explicitely

$$(6) \qquad V^1 - V^0 = (\bar{V}^1 - \bar{V}^0) - (X^1 - X^0) ,$$

where superscripts denote the two situations (e.g. periods) which are compared. In order to write this equation in terms of log-changes of the variables we use the representation

$$(7) \qquad \log(\frac{y}{x}) = \frac{y - x}{L(x,y)}$$

of the log-change as a absolute change in respect to the <u>logarithmic mean</u>

$$(8) \qquad L(x,y) = \begin{cases} \frac{y - x}{\log(y/x)}, & x \neq y \\ x & , x = y \end{cases}$$

of positive variables x and y introduced already by Törnqvist (1935, p. 35). Vartia (1976, p. 11) proves that it is really a mean of x and y and calls it the logarithmic mean. For all positive and unequal x and y the logarithmic mean satisfies

$$(9) \qquad \sqrt{xy} < T(x,y) < L(x,y) < S(x,y) < \tfrac{1}{2}(x + y) ,$$

where $T(x,y) = \sqrt[3]{xy \frac{1}{2}(x + y)}$ is the Theil (1973) mean and $S(x,y) = \frac{1}{3}(2\sqrt{xy} + \frac{1}{2}(x + y)$ is the Sato (1974) mean. If $x = y$ all means of (9) are, of course, equal. Multiplying and dividing as in Vartia (1976, p. 124) by logarithmic means we get from (6)

$$(9) \qquad \frac{V^1 - V^0}{L(V^1,V^0)} = \frac{L(\bar{V}^1,\bar{V}^0)}{L(V^1,V^0)} \left(\frac{\bar{V}^1 - \bar{V}^0}{L(\bar{V}^1,\bar{V}^0)} \right) - \frac{L(X^1,X^0)}{L(V^1,V^0)} \left(\frac{X^1 - X^0}{L(X^1,X^0)} \right) ,$$

which may be written by (8) equivalently as

$$(10) \qquad \log \left(\frac{V^1}{V^0}\right) = \frac{L(\bar{V}^1,\bar{V}^0)}{L(V^1,V^0)} \log \left(\frac{\bar{V}^1}{\bar{V}^0}\right) - \frac{L(X^1,X^0)}{L(V^1,V^0)} \log \left(\frac{X^1}{X^0}\right) .$$

It is essential that also $V = \bar{V} - X$ is necessary positive in order that $L(V^1,V^0)$ and $\log(V^1/V^0)$ are well-defined. Denoting e.g. the logarithmic mean $L(V^1,V^0)$ by $\hat{V}$ we write (10) more conveniently as

$$(11) \qquad \Delta \log V = \left(\frac{\hat{\bar{V}}}{\hat{V}}\right)\Delta \log \bar{V} - \left(\frac{\hat{X}}{\hat{V}}\right)\Delta \log X .$$

Here

$$(12) \qquad \frac{\hat{\bar{V}}}{\hat{V}} = \frac{L(\bar{V}^1,\bar{V}^0)}{L(V^1,V^0)} \approx \frac{\frac{1}{2}(\bar{V}^1 + \bar{V}^0)}{\frac{1}{2}(V^1 + V^0)} = \frac{\bar{V}^1 + \bar{V}^0}{V^1 + V^0}$$

and

$$(13) \qquad \frac{\hat{X}}{\hat{V}} = \frac{L(X^1,X^0)}{L(V^1,V^0)} \approx \frac{\frac{1}{2}(X^1 + X^0)}{\frac{1}{2}(V^1 + V^0)} = \frac{X^1 + X^0}{V^1 + V^0}$$

are a kind of mean "value shares" of $\bar{V}$ and X of $V = \bar{V} - X$, respectively. Because $\bar{V} > X$ and $V = \bar{V} - X < \bar{V}$ we have

$$(14) \qquad 0 < \frac{\hat{X}}{\hat{V}} < \frac{\hat{\bar{V}}}{\hat{V}}$$

necessarily. Also $1 < (\hat{\bar{V}}/\hat{V})$ and

$$(15) \qquad \frac{\hat{\bar{V}}}{\hat{V}} - \frac{\hat{X}}{\hat{V}} \approx \frac{\bar{V}^1 + \bar{V}^0}{V^1 + V^0} - \frac{X^1 + X^0}{V^1 + V^0} = \frac{V^1 + V^0}{V^1 + V^0} = 1 .$$

Here $(\hat{X}/\hat{V})$ may be also greater than unity. Usually the first term in equation (15) is somewhat greater than unity (i.e:, $\hat{\bar{V}} - \hat{X} > \hat{V}$), because the logarithmic mean satisfies

$$(16) \qquad \begin{aligned} L(\bar{V}^1,\bar{V}^0) &= L(V^1 + X^1, V^0 + X^0) \\ &\geqq L(V^1,V^0) + L(X^1,X^0), \end{aligned}$$

or in the notation of (15) $\hat{\bar{V}} \geq \hat{V} + \hat{X}$ or $\hat{\bar{V}} - \hat{X} \geq \hat{V}$

with equality only if $X^1/V^1 = X^0/V^0$, see Vartia (1976, p. 185-7). But (15) holds in practical situations very accurately.

Example. Consider the manufacturing of paper and paper products (Branch 341 in Industrial statistics of Finland). In the next table the value added $\bar{V}$, the gross value of production $\bar{V}$ and the value of the intermediate material input X (in mill. Fmk) are given in 1970 and 1975.

	V	$\bar{V}$	X
1970	1946	5995	4049
1975	3739	13577	9838

Here

(17) $\quad \hat{V} = L(V^1,V^0) = 2745.6 < 2843.5 = \frac{1}{2}(V^1 + V^0)$

(18) $\quad \hat{\bar{V}} = L(\bar{V}^1,\bar{V}^0) = 9275.2 < 9786.0 = \frac{1}{2}(\bar{V}^1 + \bar{V}^0)$

(19) $\quad \hat{X} = L(X^1,X^0) = 6520.7 < 6944.5 = \frac{1}{2}(X^1 + X^0)$

and the logarithmic means are considerable smaller than the arithmetic means, because the values in 1975 are more than two times those in 1970. Of course (9) would give much tighter limits for the logarithmic mean, e.g. for $\bar{V}$ ve have

(20) $$\sqrt{\bar{V}^1\bar{V}^0} < T(\bar{V}^1,V^0) < L(\bar{V}^1,\bar{V}^0) < S(\bar{V}^1,\bar{V}^0) < \frac{1}{2}(\bar{V}^1 + \bar{V}^0).$$

9021.9   9269.7   9275.2   9276.6   9786.0

We have also

$$(21) \qquad \hat{\bar{V}} = 9275.6 < 9266.3 = \hat{V} + \hat{X}$$

the relative difference of $\hat{\bar{V}} = L(\bar{V}^1, \bar{V}^0) = L(V^1 + X^1, V^0 + X^0)$ and $\hat{V} + \hat{X} = L(V^1, V^9) + L(X^1, X^0)$ being negligible, namely 0.10 %. For the difference (15) of mean value shares we get

$$(22) \qquad \frac{\hat{\bar{V}}}{\hat{V}} - \frac{\hat{X}}{\hat{V}} = 3.3782 - 2.3750 = 1.0032 > 1.0000,$$

where 1.0032 exceeds unity only by 0.32 %. In the manufacturing of paper and paper products the ratios of gross output and intermediate inputs in respect to value added (i.e., 3.3782 and 2.3750) are considerably greater than in an average branch of industry. In Finland the average values of $\hat{\bar{V}}/\hat{V}$ and $\hat{X}/\hat{V}$ are about 2.8 and 1.8. For some branches they are as low as 1.7 and 0.7.

Equation (11) says that the log-change in variable V (value of value added) is a linear combination (practically a generalized weighted average[1)]) of the log-changes in $\bar{V}$ (value of gross output) and X (value of intermediate material inputs) the weights being their mean value shares (more accurately value relatives) $\hat{\bar{V}}/\hat{V}$ and $\hat{X}/\hat{V}$ in respect to V. This equation holds for all possible values of the variables.

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1) <u>A weighted (arithmetic) average</u> of x_i's with c_i's as weights is $\Sigma\, c_i x_i / \Sigma\, c_i = \Sigma\, c_i x_i$, if $\Sigma\, c_i = 1$. Often weights c_i are assumed to be nonnegative as well, but this not necessary. If at least one of the weights c_i is negative we call $\Sigma c_i x_i / \Sigma c_i$ a <u>generalized</u> weighted average.

More symmetric and perhaps intuitive ways to write (11) are the following:

$$(23) \qquad \hat{V}\Delta\log V = \hat{\bar{V}}\Delta\log\bar{V} - \hat{X}\Delta\log X$$

or equivalently

$$(24) \qquad \hat{\bar{V}}\Delta\log\bar{V} = \hat{V}\Delta\log V + \hat{X}\Delta\log X$$

$$(25) \qquad \Delta\bar{V} = \Delta V + \Delta X ,$$

a mere triviality. Equation (24) may be written also as

$$(26) \qquad \Delta\log\bar{V} = \left(\frac{\hat{V}}{\hat{\bar{V}}}\right)\Delta\log V + \left(\frac{\hat{X}}{\hat{\bar{V}}}\right)\Delta\log X ,$$

which gives the log-change in the value of the gross output $\Delta\log\bar{V}$ as a linear combination (practically a generalized weighted average) of the log-changes $\Delta\log V$ and $\Delta\log X$ in the values of the value added and intermediate inputs. A similar equation led us to the invention of a new index number formula, see Vartia (1976, p. 123-6). Equation (11) may be derived from (26) by simply solving for $\Delta\log\bar{V}$, which show that (26) and (11) are in fact equivalent representations of the same relationship. The log-changes $\Delta\log\bar{V}$, $\Delta\log V$ and $\Delta\log X$ must satisfy both equations if they are mutually consistent.

This equation leads us in a natural way to the new double deflation type of price and quantity indices for the value added. Let

(27) $\bar{V}^1/\bar{V}^0 = \bar{P}_0^1\bar{Q}_0^1$

be a representation of the value ratio of the gross output by an arbitrary pair of price and quantity indices $(\bar{P}_0^1, \bar{Q}_0^1)$ for it. Similarly let (π_0^1, κ_0^1) be an arbitrary "dual pair" of price and quantity indices for the intermediate material inputs satisfying

(28) $X^1/X^0 = \pi_0^1\kappa_0^1$.

Taking logarithms and inserting into (10) we get

$$(29) \qquad \log \left(\frac{V^1}{V^0}\right) = \frac{\hat{\bar{V}}}{\hat{V}} \log (\bar{P}_0^1\bar{Q}_0^1) - \frac{\hat{X}}{\hat{V}} \log (\pi_0^1\kappa_0^1)$$

$$= \left(\frac{\hat{\bar{V}}}{\hat{V}} \log \bar{P}_0^1 - \frac{\hat{X}}{\hat{V}} \log \pi_0^1\right) +$$

$$\left(\frac{\hat{\bar{V}}}{\hat{V}} \log \bar{Q}_0^1 - \frac{\hat{X}}{\hat{V}} \log \kappa_0^1\right)$$

It is natural to define the pair (P_0^1, Q_0^1) of price and quantity indices of the value added by

$$(30) \qquad \log P_0^1 = \frac{\hat{\bar{V}}}{\hat{V}} \log \bar{P}_0^1 - \frac{\hat{X}}{\hat{V}} \log \pi_0^1$$

$$(31) \qquad \log Q_0^1 = \frac{\hat{\bar{V}}}{\hat{V}} \log \bar{Q}_0^1 - \frac{\hat{X}}{\hat{V}} \log \kappa_0^1 .$$

These definitions are completely symmetrical with each other and with the equation for the value of the value added

$$(32) \qquad \log\frac{V^1}{V^0} = \frac{\hat{\bar{V}}}{\hat{V}}\log\frac{\bar{V}^1}{\bar{V}^0} - \frac{\hat{X}}{\hat{V}}\log\frac{X^1}{X^0}$$

Price and quantity indices (P_0^1, Q_0^1) of the value added may be directly calculated from (24) and (25) when estimates of the price and quantity indices $(\bar{P}_0^1, \bar{Q}_0^1)$ and (π_0^1, κ_0^1) of gross output, and intermediate material inputs and of the weights $(\hat{\bar{V}}/\hat{V}, \hat{X}/\hat{V})$ are available. The log-change $\log P_0^1$ of P_0^1 is a linear combination (and practically a generalized weighted average) of the log-changes $\log\bar{P}_0^1$ and $\log\pi_0^1$ of P_0^1 and π_0^1. Similarly $\log Q_0^1$ is a linear combination of $\log\bar{Q}_0^1$ and $\log\kappa_0^1$ calculated using the same weights.
Just like (32) is equivalent to (26) equations (30) and (31) may be written equivalently as

$$(33) \qquad \log\bar{P}_0^1 = \left(\frac{\hat{V}}{\hat{\bar{V}}}\right)\log P_0^1 + \left(\frac{\hat{X}}{\hat{\bar{V}}}\right)\log\pi_0^1$$

$$(34) \qquad \log\bar{Q}_0^1 = \left(\frac{\hat{V}}{\hat{\bar{V}}}\right)\log Q_0^1 + \left(\frac{\hat{X}}{\hat{\bar{V}}}\right)\log\kappa_0^1 ,$$

which reveal nicely the nature of the value added indices P_0^1 and Q_0^1. For instance the price index P_0^1 is defined by (30) in such a way that the price index $\bar{P}_0^1$ of the gross output is a Vartia price index (see Vartia (1976, p. 123-6)) calculated from the price indices of P_0^1 and π_0^1, which are the input price indices of labour/capital services (P_0^1) and intermediate material inputs (π_0^1). The weights $\hat{V}/\hat{\bar{V}}$ and $\hat{X}/\hat{\bar{V}}$ are a kind of values shares of the value added V and the intermediate material inputs X of the gross output $\bar{V}$. Of course the value shares $\hat{V}/\hat{\bar{V}}$ and $\hat{X}/\hat{\bar{V}}$ appearing symmetrically in all the equations (26), (33)-(34) are both positive, which in turn implies that the common weight of $\log \pi_0^1$, $\log \kappa_0^1$ and $\log(X^1/X^0)$ in (30)-(32) must be negative, i.e. $-(\hat{X}/\hat{V})$.

The Vartia price and quantity indices $(P_0^1(V), Q_0^1(V))$ defined for the general case of n commodities $a_1,\ldots,a_n$ (say consumer goods) are based on the representation

$$(35)\qquad \log\left(\frac{V^1}{V^0}\right) = \log\left(\sum_{i=1}^{n} v_i^1 / \sum_{i=1}^{n} v_i^0\right)$$

$$= \sum_{i=1}^{n} \frac{L(v_i^1, v_i^0)}{L(V^1, V^0)} \log\left(\frac{v_i^1}{v_i^0}\right)$$

of the log-change in total value $V = \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} p_i q_i$ in terms of the log-changes in the values $v_i = p_i q_i$ of individual commodities:

$$(36)\qquad \log P_0^1(V) = \sum_{i=1}^{n} \frac{L(v_i^1, v_i^0)}{L(V^1, V^0)} \log \pi_0^1(i)$$

$$(37)\qquad \log Q_0^1(V) = \sum_{i=1}^{n} \frac{L(v_i^1, v_i^0)}{L(V^1, V^0)} \log \kappa_0^1(i)\ ,$$

where $\pi_0^1(i) = p^1/p_i^0$ and $\kappa_0^1(i) = q_i^1/q_i^0$. Clearly $(P_0^1(V), Q_0^1(V))$ satisfy the strong factor reversal test, $V^1/V^0 = P_0^1(V)Q_0^1(V)$, and the time reversal test, $P_1^0(V) = 1/P_0^1(V)$ and $Q_1^0(V) = 1/P_0^1(V)$. For other nice properties of the Vartia index, see Vartia (1976, 1976b) and Diewert (1978). The sum of weights $L(v_i^1, v_i^0)/L(V^1, V^0)$ in (35)-(37) will be somewhat smaller than unity (unless all values have changed proportionally: v_i^1/v_i^0 = constant), but this does not lead to other substantial problems except perhaps psychological ones, see Vartia (1976, p. 126 and 140, 1976b). The logarithms of Vartia indices (36) and (37) are practically weighted arithmetic averages of the price and quantity log-changes, i.e. ($P_0^1(V)$ and $Q_0^1(V)$ may be considered as weighted geometric averages of $\pi_0^1(i)$'s and $\kappa_0^1(i)$'s, respectively.

3. Comparing the new double deflation method with the tradional one

It is illustrative the compare the new price and quantity indices of the value added (P_0^1, Q_0^1) defined by (30)-(31) with the tradional "double deflation indices" or Geary indices of value added, see Hill (1971). As mentioned in the introduction the tradional double deflation method of defining the volume of value added just uses its value in constant prices calculated as the difference between the value of the gross output and total intermediate inputs, each measured in constant prices. In our previous notation the value of gross output in constant prices is $\bar{V}^0\bar{Q}_0^1$ and the value of total intermediate inputs in constant prices is $X^0\kappa_0^1$. Their difference $\bar{V}^0\bar{Q}_0^1 - X^0\kappa_0^1$ is taken here as the indicator of value added in constant prices, i.e.,

$$(38) \qquad V^0Q_0^1 = \bar{V}^0\bar{Q}_0^1 - X^0\kappa_0^1 .$$

Dividing by the base year value V^0 of the value added we get the <u>Geary quantity index of the value added</u>

$$(39) \qquad Q_0^1 = \frac{\bar{V}^0\bar{Q}_0^1 - X^0\kappa_0^1}{V^0} =$$

$$= \frac{\bar{V}^0}{V^0}\bar{Q}_0^1 - \frac{X^0}{V^0}\kappa_0^1 = \bar{Q}_0^1 + \frac{X^0}{V^0}(\bar{Q}_0^1 - \kappa_0^1)$$

Geary (1944) seems to be the first who has used this method. The second equation shows that Q_0^1 is a generalized weighted average of $\bar{Q}_0^1$ and κ_0^1 the weights being $(\bar{V}^0/V^0)$ and $-(X^0/V^0)$. The sum of weights equals unity: $(\bar{V}^0/V^0) - (X^0/V^0) = (\bar{V}^0 - X^0)/V^0 = V^0/V^0 = 1$. Because the weight $-(X^0/V^0)$ of the quantity index κ_0^1 of the intermediate inputs is negative the Geary volume index Q_0^1 of the value added never lies between $\bar{Q}_0^1$ and κ_0^1, as Hill (1971, p. 17) shows. This may be at the first sight rather surprising. However, this is just how the things should be, because value added is the difference between the gross output and intermediate inputs. By solving (39) for the quantity index $\bar{Q}_0^1$ of the gross output we get

$$(40) \qquad \bar{Q}_0^1 = \frac{V^0}{\bar{V}^0} Q_0^1 + \frac{X^0}{\bar{V}^0} \kappa_0^1 ,$$

an ordinary weighted average of Q_0^1 and κ_0^1 with corresponding proper value shares of gross output as weights, cf. (34). This would be just Laspeyres' ordinary method of aggregating Q_0^1 and κ_0^1 into $\bar{Q}_0^1$ if Q_0^1 were known. Therefore, the Geary quantity index Q_0^1 of value added is defined in such a way that the weighted average of it and the given quantity index κ_0^1 of intermediate inputs equals the given quantity index $\bar{Q}_0^1$ of gross output.

From equation (40) we see at once that $\bar{Q}_0^1$ always lies between Q_0^1 and κ_0^1, a very natural requirement. Therefore the equivalent reguirement that Q_0^1 must not lie between $\bar{Q}_0^1$ and κ_0^1 is equally natural. From equation (34) we see in a similar way that it is quite natural that $\log Q_0^1$ (where Q_0^1 is now our new volume index of value added) does not lie between $\log\bar{Q}_0^1$ and $\log \kappa_0^1$, or equivalently that $Q_0^1 \leq \min(\bar{Q}_0^1, \kappa_0^1)$ or $\max(\bar{Q}_0^1, \kappa_0^1) \leq Q_0^1$.

The price index P_0^1 of the value added related to the choise (39) is defined implicitely by $P_0^1 Q_0^1 = V^1/V^0$. Therefore this P_0^1 satisfies

$$(41) \qquad P_0^1 = \frac{V^1/V^0}{\frac{\bar{V}^0}{V^0}\bar{Q}_0^1 - \frac{X^0}{V^0}\kappa_0^1}$$

$$= \frac{V^1}{\bar{V}^0\bar{Q}_0^1 - X^0\kappa_0^1}$$

$$= \frac{V^1}{\bar{V}^0\frac{\bar{P}_0^1\bar{Q}_0^1}{\bar{P}_0^1} - X^0\frac{\Pi_0^1\kappa_0^1}{\Pi_0^1}}$$

$$= \frac{V^1}{\frac{\bar{V}^1}{\bar{P}_0^1} - \frac{X^1}{\Pi_0^1}},$$

because $\bar{P}_0^1\bar{Q}_0^1 = \bar{V}^1/\bar{V}^0$ and $\kappa_0^1\Pi_0^1 = X^1/V^0$. Thus equivalently

$$(42) \qquad (P_0^1)^{-1} = \frac{\bar{V}^1}{V^1}(\bar{P}_0^1)^{-1} - \frac{X^1}{V^1}(\Pi_0^1)^{-1}$$

and P_0^1 is a generalized weighted harmonic mean of $\bar{P}_0^1$ and π_0^1 with weights $(\bar{V}^1/V^1)$ and $-(X^1/V^1)$ the sum of which is unity. While the Geary quantity index (39) of value added is a generalized weighted arithmetic mean of $\bar{Q}_0^1$ and κ_0^1 (the weights being from the base period) the corresponding (implicite) price index (41) of value added is <u>not</u> of the same type, but a generalized weighted harmonic mean of $\bar{P}_0^1$ and Π_0^1 (the weights being from the current period). This is an unsymmetric feature in Geary's double deflation method. In our new method given by equations (30)-(31) the definitions of price and quantity indices of value added are completely symmetrical.

As in the case of quantity index also the equation (41) of the price index may be solved for $\bar{P}_0^1$ as follows

$$(43) \qquad (\bar{P}_0^1)^{-1} = \frac{V^1}{\bar{V}^1}(P_0^1)^{-1} + \frac{X^1}{\bar{V}^1}(\Pi_0^1)^{-1} .$$

This gives the intuitive meaning of P_0^1: it is the unique price index of value added which weighted together with the given price index Π_0^1 of the intermediate inputs gives the given price index $\bar{P}_0^1$ of the gross output. We recognize that $\bar{P}_0^1$ of (43) appears as the Paasche index of P_0^1 and Π_0^1, while in (40) $\bar{Q}_0^1$ is related to Q_0^1 and $\;_0^1$ by Laspeyres' formula. This is another way to reveal the asymmetric character of Geary's double deflation method. Note that, just like in the case of quantity index (39) the price index (41) must <u>not</u> lie between $\bar{P}_0^1$ and Π_0^1.

A serious difficulty with Geary's price and quantity indices of value added is that they may get negative values. As Hill (1971, p. 19) notes by passing Q_0^1 of (39) may be negative and even approach minus infinity if $\bar{Q}_0^1 < \kappa_0^1$ and $\bar{V}^0/V^0$ is is considerably greater than unity (i.e., the value added V^0 is just a small part of the gross product $\bar{V}^0$). David (1962) considers the same problem more carefully. If Q_0^1 is negative then also the price index P_0^1 of the value added is negative because their product $P_0^1 Q_0^1$ equals the value ratio V^1/V^0 of the value added, which was supposed to be positive. In this respect our new price and quantity indices of value added defined by (30)-(31) behave in a completely safe way. Although their logarithms may well be negative the indices themselves are always positive.

By interchanging price and quantity indices in the definitions (39) and (42) we get another "dual pair" (P_0^1, Q_0^1) of value added indices:

$$(44) \qquad P_0^1 = \frac{\bar{V}^0}{V^0} \bar{P}_0^1 - \frac{X^0}{V^0} \pi_0^1$$

$$(45) \qquad (Q_0^1)^{-1} = \frac{\bar{V}^1}{V^1} (\bar{Q}_0^1)^{-1} - \frac{X^1}{V^1} (\kappa_0^1)^{-1} .$$

The indices P_0^1 and Q_0^1 may be defined equivalently as solutions to a more familiar pair of equations

$$(46) \qquad \bar{P}_0^1 = \frac{V^0}{\bar{V}^0} P_0^1 + \frac{X^0}{\bar{V}^0} \pi_0^1 \qquad \text{"Laspeyres"}$$

$$(47) \qquad (\bar{Q}_0^1)^{-1} = \frac{V^1}{\bar{V}^1} (Q_0^1)^{-1} + \frac{X^1}{\bar{V}^1} (\kappa_0^1)^{-1} . \qquad \text{"Paasche"}$$

Indices P_0^1 and Q_0^1 above have the same asymmetric features as the Geary indices and they may get as well negative values. We have presented them to demonstrate that numerous indices of the "double deflation type" may be derived starting from other well-known index number formulas instead of previous Laspeyres' and Paasche's formulas. Although most of the resulting new "double deflation indices" are unpleasantly complicated some formulas give almost as simple results as above. For instance, take instead of Paasche's formula (47) Törnqvist I-formula, see Vartia (1976, p. 135, 203):

$$(48) \qquad \log\bar{Q}_0^1 = \left(\frac{V^1 + V^0}{\bar{V}^1 + \bar{V}^0}\right) \log Q_0^1 + \left(\frac{X^1 + X^0}{\bar{V}^1 + \bar{V}^0}\right) \log \kappa_0^1 .$$

This is easily solved for $\log Q_0^1$

$$(49) \qquad \log Q_0^1 = \left(\frac{\bar{V}^1 + \bar{V}^0}{V^1 + V^0}\right) \log\bar{Q}_0^1 - \left(\frac{X^1 + X^0}{V^1 + V^0}\right) \log \kappa_0^1$$

and the weights are of the same simple type as in (31). In fact (49) and (31) lead to almost identical quantity indices Q_0^1 of value added if value ratios V^1/V^0, X^1/X^0 and $\bar{V}^1/\bar{V}^0$ do not differ much from each other. This holds because Törnqvist I-formula (48) and Vartia-formula (34) are for moderate relative changes good approximations of each other. Hill (1971, p. 19) points out that,

> "in general, the index of value added obtained by double deflation is more sensitive to error than either of the two indices from which it is derived. On the other hand, a single indicator based on output (or input) alone gives a biassed estimate of movements in real product as it fails to take account of any real divergence between the movements of input and output".

The same applies qualitatively to any double deflation index and not only to Geary's quantity index (39), which Hill expresses in the form

$$(50) \qquad Q_0^1 = \lambda\bar{Q}_0^1 - (\lambda-1)\,\kappa_0^1 = \lambda\xi - (\lambda-1)\gamma$$

where $\lambda = \bar{V}^0/V^0$. Let

$$(51) \qquad Q_0^1(c) = c\bar{Q}_0^1 - (c-1)\,\kappa_0^1$$

be a (generalized) weighted arithmetic average of $\bar{Q}_0^1$ and κ_0^1. Then for instance $Q_0^1(\lambda) = Q_0^1$, $Q_0^1(1) = \bar{Q}_0^1$ and $Q_0^1(0) = \kappa_0^1$. We refer to Hill's profound analysis how the weight c in (51) should be chosen when both $\bar{Q}_0^1$ and κ_0^1 contain sampling or other measurement errors.

4. Tentative calculations

We illustrate the different methods of calculating the price and quantity indices of value added by more or less unreliable data related to manufacturing of paper and paper products (Branch 341) in Finland in 1970-75. The values and value ratios of gross product ($\bar{V}^t$), intermediate inputs (X^t) and value added (V^t) of Branch 341 are given in the following table.

Table 1. Manufacturing of paper and paper products (Branch 341) in Finland in 1970-75, mill. mk

Year	Gross product		Intermediate inputs		Value added	
t	$\bar{V}^t$	$\bar{V}^t/\bar{V}^{1970}$	X^t	X^t/X^{1970}	V^t	V^t/V^{1970}
1970	5 995	1.000	4 049	1.000	1 946	1.000
1971	6 472	1.086	4 745	1.172	1 727	0.887
1972	7 352	1.226	5 375	1.327	1 977	1.016
1973	8 794	1.460	6 297	1.555	2 497	1.283
1974	13 600	2.269	9 458	2.336	4 142	2.128
1975	13 577	2.265	9 838	2.430	3 739	1.921

Source: Industrial statistic, Official statistics in Finland.

There is a notable difference in the value ratio of gross product and intermediate inputs particularly in 1971. Therefore the development of the value ratio V^t/V^{1970} of the value added differs quite much of movements in output and input. Rather crude estimates of the price indices of gross product $\bar{P}^t_{1970}$ and intermediate inputs Π^t_{1970} were calculated using various sources. These and the quantity indices $\bar{Q}^t_{1970}$ and κ^t_{1970} calculated by deflating the value ratios by corresponding price indices are given in table 2.

Table 2. Estimated price and quantity indices of gross product and intermediate inputs in branch 341 in 1970-75.

year	Gross product		Intermediate inputs	
t	$\bar{P}^t_{1970}$	$\bar{Q}^t_{1970}$	Π^t_{1970}	κ^t_{1970}
1970	1.000	1.000	1.000	1.000
1971	1.076	1.009	1.108	1.058
1972	1.086	1.129	1.143	1.161
1973	1.175	1.243	1.543	1.008
1974	1.705	1.331	2.123	1.100
1975	2.361	0.961	2.294	1.059

The prices of input in 1973-74 were considerably higher than output prices, but in 1975 both input and output prices were about 2.3-fold compared to those of 1970. Quantity indices $\bar{Q}^t_{1970}$ and κ^t_{1970} differ considerably also in 1973-74. Therefore different methods of calculating price and quantity indices of value added will produce different results particularly in 1973 and 1974.

We represent estimates only for the quantity index of value added, because the corresponding dual price index may be calculated simply by division, $P^t_{1970} = \dfrac{V^t/V^{1970}}{Q^t_{1970}}$.

In table 3 four members of the family $Q^t_{1970}(c)$ of (51) and formulas Q^t_{1970} of (45) and (49) are presented.

Table 3. Quantity indices of the value added in branch 341 calculated using different methods

year	(1)	(2)	(3)	(4)	(5)	(6)
t	$Q^t_{1970}(3.081)$	$Q^t_{1970}(1)$	$Q^t_{1970}(0)$	$Q^t_{1970}(0.5)$	Q^t_{1970} of (45)	Q^t_{1970} of (49)
1970	1.000	1.000	1.000	1.000	1.000	1.000
1971	0.907	1.009	1.058	1.034	0.895	0.901
1972	1.062	1.129	1.161	1.145	1.050	1.056
1973	1.732	1.243	1.008	1.126	3.017	2.025
1974	1.812	1.331	1.100	1.216	2.557	2.032
1975	0.757	0.961	1.059	1.010	0.773	0.758

The official quantity index of the value added gives essentially the same results as $Q^t_{1970}(1) = \bar{Q}^t_{1970}$, which is a single indicator based on output only. The official index is also based essentially on the development of physical output. Columns (1), (5) and (6) represent double deflation indices. In column (1) the figures of Geary's quantity index are given and they differ considerably[1] from figures of (2) and (3) lying never between them as the theory requires. In column (5) a Paasche's type of quantity index of the value added is presented; the figures of columns (1) and (2) differ considerably in 1973 and 1974. The Törnqvist's type of quantity index of the value added is shown in column (6) and it gives results always near the geometric mean of (1) and (5). Note also that figures of (5) and (6) cannot lie between those of (2) and (3). If Q^t_{1970} were calculated also using our favourite formula (31) results would practically coincide with column (6) the difference being also in the difficult year 1973-74 less than 1 %.

The development of the quantity of value added is thus affected in a significant way by the choice of calculating method of the index. Especially indices of the double deflation type differ systematically from indices based on output or input only and also from any indices (like that of column (4)), which are their proper averages. Substantial differences may occur also between indices of the double deflation type. The reliability of the price, quantity and value data is an essential factor which determines whether

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1) After making the calculations it became evident that the official value series of gross product and intermediate inputs in manufacturing of paper and paper products include some serious double counting, see Ylä-Anttila (1978, p. 20). We have presented the calculations based on the official but misleading figures as a warning example.

a double deflation method can be recommended. If the input data allows the use of some double deflation method then our new symmetric type of double deflation indices or their good approximations should be used rather than the former asymmetric double deflation methods.

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