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FISHER'S FIVE-TINED FORK AND OTHER QUANTUM THEORIES OF INDEX NUMBERS

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1. INTRODUCTION

Among Fisher's (1922) most interesting contributions are his propositions concerning the biases of index number formulas. Weighted index numbers (omitting modes and medians) seem to cluster into five groups according to the type of the average and the weights used. Fisher explains this using the concepts of 'type bias' and 'weight bias' interacting with each other, see Fisher (1922) p. 83-117 and 352-6. His theory is condensed in a graphical representation, called the Five-tined Fork, each tine representing index numbers having the same 'dose of bias', i.e., 2+, 1+, 0, 1- or 2-. For instance the group 2+ consists of weighted index numbers (except modes and medians) having a double upward bias, see Fisher (1922) p. 202-205.

Fisher concludes on p. 204-5:

"Thus, barring 'simples' and 'modes' and their derivates (and possibly medians if we wish to have our results very close), we find that, although we have numerous formulae, they all fall under only five clearly defined heads, namely, those without bias, those with single bias up or down, and those with double bias up or down.

The five tines include all the arithmetic, harmonic, geometric, and aggregative weighted index numbers and their derivates which we have obtained."

Fisher's 'Five-tined Fork' may be well described as a 'quantum theory' of index numbers to distinguish it from an ordinary view,

* I want to express my gratitude to my teacher, prof. Leo Törnqvist for numerous stimulating conversations and to the participants of the symposium for valuable comments. Jaakko Railo, M.A., has checked my English. All remaining errors are mine.

according to which the results of various index formulas disperse continuously without gaps making a broom-like picture.

Let $a_1, \ldots, a_n$ be n commodities or groups of commodities for which the indices will be defined. Denote the value of a_i by v_i (in money units), its quantity by q_i (in physical units), price by $p_i = v_i/q_i$ and value share by $w_i = v_i/\Sigma v_j$. Periods or places are indicated by superscripts 0, 1 etc. Price and quantity vectors are denoted p and q, $p \cdot q = \Sigma p_i q_i$ is their inner product.

As a summary of Fisher's findings we consider the following price[1] index number formulas

(1) $L = p^1 \cdot q^0 / p^0 \cdot q^0 = \Sigma w_i^0 (p_i^1/p_i^0)$, "Laspeyres"

(2) $P = p^1 \cdot q^1 / p^0 \cdot q^0 = 1/\Sigma w_i^1 (p_i^0/p_i^1)$, "Paasche"

(3) $F = \sqrt{L \cdot P}$, "Fisher"

(4) $\log l = \Sigma w_i^0 \log (p_i^1/p_i^0)$, "Logarithmic Laspeyres"

(5) $\log p = \Sigma w_i^1 \log (p_i^1/p_i^0)$, "Logarithmic Paasche"

(6) $\log t = \frac{1}{2}(\log l + \log p)$, "Törnqvist"

(7) $Pl = \Sigma w_i^1 (p_i^1/p_i^0)$, "Palgrave"

(8) $Lh = 1/\Sigma w_i^0 (p_i^0/p_i^1)$, "Harmonic Laspeyres"

They are classified in Fisher's five tines as follows, see Fisher (1922) p. 204.

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1) We need not consider quantity index number formulas separately because everything applies analogically to them after changing p_i:s and q_i:s.

Table 1. Fisher´s Five-tined Fork

Tine	Formula	Fisher's corresponding symbols
Uppermost (2+)	Pl	9
Mid-upper (1+)	log p	29
Middle (0)	P F, log t L	4=5=18=19=54=59 353[1], 123 3=6=17=20=53=60
Mid-lower (1-)	log l	23
Lowermost (2-)	Lh	13

The results of Fisher´s calculations are presented in the following table.

Table 2. The results of Fisher´s calculations

Index number formula	Year					
	1913	1914	1915	1916	1917	1918
Pl, 9	100	100.93	102.33	118.29	180.72	187.18
logp, 29	100	100.63	101.17	116.26	170.44	182.41
P, 54	100	100.32	100.10	114.35	161.05	177.43
F, 353	100	100.12	99.89	114.21	161.56	177.65
logt, 123	100	100.12	99.94	113.83	162.05	177.80
L, 53	100	99.93	99.67	114.08	162.07	177.87
logl, 23	100	99.61	98.72	111.45	154.08	173.30
Lh, 13	100	99.26	97.84	111.01	147.19	168.59

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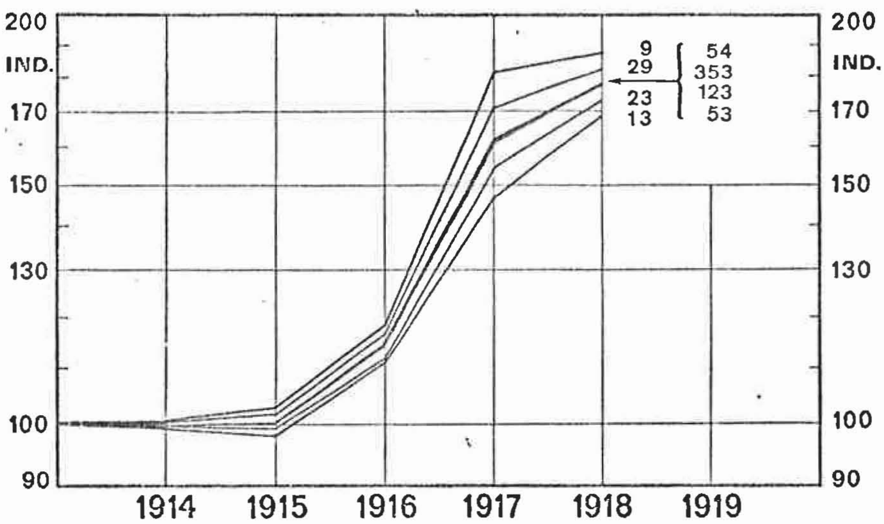
1) Fisher's Ideal index F may be defined in numerous different ways, which is shown by its other symbols 103, 104, 105, 106, 153, 154, 203, 205, 217, 219, 253, 259, 303 and 305. This means that F has many fruitful interpretations; it is not just 'the geometric mean of L and P'.

Figure 1: Fisher's Five-tined Fork for 8 Price Indices



According to his calculations Fisher finds that L (=54) and P (=53) give approximately the same results and classifies them to the group 0 of unbiased index numbers. On the contrary log p (=29) and log l (=23) seem to contain respectively a single upward and downward bias. As Fisher concludes on p. 363:

"Of the 25 formulae mentioned by previous writers as possibly valuable, we have seen that the following ought never be used because of bias: 1, 2, 9, 11, 23." And on p. 364 he writes:

"Thus as to the long controversy over the relative merits of the arithmetic and geometric types, our study shows us that the _simple_ geometric, 21, is better than the simple arithmetic, 1, but that, curiously enough, the _weighted_ arithmetic, 3, is better than the weighted geometric, 23."

Fisher (see p. 237) did not regard the close agreement of L and P as "an accident merely happening to be true for the 36 commodities selected". Fisher admits on p. 239-240 and 410 that L and P are subject to a "sort of secondary bias", which he regarded, however, as very small. We will show that these conclusions of Fisher are based on an unwarranted belief of the representativeness of his data and are not generally valid. For some other data his inductive reasoning would have given other results.

Our analysis fits in with what has been pointed out by other authors. For instance Samuelson & Swamy (1974) p. 567 comment on Fisher's concept of bias: "Exactly what zero bias meant was never thought through." The well-known inequalities connected with Laspeyres' and Paasche's indices show that these are clearly biased respectively upwards and downwards as compared to the 'true indices' in the case of demand theory:

(9) $$P(p^1,p^0;\ q^0) \leq p^1\cdot q^0/p^0\cdot q^0 = L_p$$

(10) $$P(p^1,p^0;\ q^1) \geq p^1\cdot q^1/p^0\cdot q^1 = P_p$$

(11) $$Q(q^1,q^0;\ p^0) \leq p^0\cdot q^1/p^0\cdot q^0 = L_q$$

(12) $$Q(q^1,q^0;\ p^1) \geq p^1\cdot q^1/p^1\cdot q^0 = P_q$$

Here $P(p^1,p^0;\ q^*)$ is the Economic Price Index and $Q(q^1,q^0;\ p^*)$ is the Economic Quantity Index as defined by Samuelson & Swamy.
In the case of production theory the inequalities are reversed, see Samuelson & Swamy (1974) p. 589 and Fisher & Shell (1972) p. 58. Only if q^0 and q^1 are indifferent or the indifference surfaces are homothetic are the Economic Price Indices in (9) and(10) equal and

we have a double inequality for it. Analogously, only if $p^0 = \lambda p^1$ or under homotheticity have we necessarily
$P_q \leq Q(q^1,q^0; p^1) = Q(q^1,q^0; p^0) \leq L_q$.
It is difficult to understand that these bounds have given rise to so much confusion. Nice examples of the kind of confusion are given e.g. by Leontief (1936) on p. 47 and by Frisch (1936) on p. 26.

On the other hand it can be shown[1)] that log p and log l are linear approximations to log $P(p^1,p^0; q^0)$ and log $P(p^1,p^0; q^1)$ in the case of demand theory.

2. EXPLANATION OF FISHER'S FIVE-TINED FORK AND OTHER QUANTUM THEORIES OF INDEX NUMBERS

These facts suggest that the situation is not so simple as Fisher thought. We are not, however, satisfied with these results of the economic approach: they are valid only if our data is generated according to some economic play process, e.g., the demand theory. We want to know how much and why the various price and volume indices differ when prices and quantities 'change freely', i.e., in any way whatsoever. We have calculated relative differences between various indices using a formula given by Törnqvist (1936).

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1) Rajaoja (1958) proves only that log l = log$P(p^1,p^0; q^0)$+ second order terms in her theorem 8.3. where she makes unrealistic assumptions about observations. The change of real income between observations (p^0,q^0), (p^1,q^1) should be taken into account e.g. in the way Theil (1967) p. 216 does.

Törnqvist considers equally weighted moment means ${}_{\alpha}P_0^1$ and geometric means ${}_0P_0^1$ of price ratios defined by

$$(13) \qquad ({}_{\alpha}P_0^1)^{\alpha} = \Sigma c_i (p_i^1/p_i^0)^{\alpha} = \Sigma c_i \; e^{\alpha \log (p_i^1/p_i^0)}$$

$$(14) \quad \log({}_0P_0^1) = \Sigma c_i \log(p_i^1/p_i^0) \quad ,$$

where $c_i \geq 0$ and $\Sigma c_i = 1$. It may be shown that the moment mean ${}_{\alpha}P_0^1$ of positive and nonequal price ratios is a continuously increasing function of α, which approaches $\min(p_i^1/p_i^0)$ when $\alpha \to -\infty$, the geometric mean ${}_0P_0^1$ defined by (14) when $\alpha \to o$ and $\max(p_i^1/p_i^0)$ when $\alpha \to +\infty$, see Hardy & Littlewood & Polya (1952).

Dividing every term of (13) by $({}_0P_0^1)^{\alpha}$ we get

$$(15) \qquad ({}_{\alpha}P_0^1/{}_0P_0^1)^{\alpha} = \Sigma c_i \; e^{\alpha \log (p_i^1/p_i^0 {}_0P_0^1)}$$

$$= \Sigma c_i \; e^{\alpha \dot{p}_i} \quad ,$$

where $\dot{p}_i = \log(p_i^1/p_i^0) - \log({}_0P_0^1)$ is the logarithmic deviation[1)] of the price ratio from ${}_0P_0^1$. By expanding (15) to a power series of α we get for all values of $\dot{p}_i$:s

$$(16) \qquad ({}_{\alpha}P_0^1/{}_0P_0^1)^{\alpha} = 1 + \frac{\alpha^2}{2!}\Sigma c_i \dot{p}_i^2 + \frac{\alpha^3}{3!}\Sigma c_i \dot{p}_i^3 + \cdots \; .$$

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1) Or the arithmetic deviation of the log-change in the price of commodity a_i from the log-change in the price level.

Taking logarithms and expanding we get formally[1)]

(17) $$\log({}_{\alpha}P_0^1) - \log({}_0P_0^1) = \frac{\alpha}{2}\, s_p^2 + \frac{\alpha^2}{6}\Sigma c_i \dot{p}_i^3 + \cdots ,$$

where $s_p^2 = \Sigma c_i \dot{p}_i^2$ is the variance of the price log-changes $\log(p_i^1/p_i^0)$ around their mean $\log({}_0P_0^1)$, shortly 'variance of the price changes'. Specifying $\alpha = 1$ and $\alpha = -1$ and neglecting the higher order terms we get:

(18) $$\log({}_1P_0^1) - \log({}_0P_1^1) \approx + \frac{1}{2}\, s_p^2 + \frac{1}{6}\, \Sigma c_i \dot{p}_i^3$$

(19) $$\log({}_{-1}P_0^1) - \log({}_0P_1^1) \approx - \frac{1}{2}\, s_p^2 + \frac{1}{6}\, \Sigma c_i \dot{p}_i^3 .$$

These express that the arithmetic mean ${}_1P_0^1 = \Sigma c_i (p_i^1/p_i^0)$ is greater than the geometric mean ${}_0P_0^1$, which is greater than the harmonic mean ${}_{-1}P_0^1$, ${}_1P_0^1 > {}_0P_0^1 > {}_{-1}P_0^1$, their logarithmic differences being approximately half of the variance of the price changes s_p^2. This is the mathematical basis for a quantitative version of Fisher's qualitative and partly inductive theory about the 'type bias' of index number formulas, cf. Fisher (1922) p. 83-91 and 108-111. Although Fisher treated the 'type bias' correctly his inductive reasoning led him to incorrect generalizations in the case of 'weight bias' as we skall demonstrate.

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1) The expansion is valid if the right hand side of (16) does not exceed 2. This is certainly true if $|\alpha \dot{p}_i| < \log 2 = 0.693$ for all i. In most practical cases (17) is valid. Note that the first term of the expansion always gives the right sign for the left side difference and they are zero simultaneously.

Using the weights $c_i = w_i^1$ we get the logarithmic differences between (2), (5) and (7):

$$(20) \qquad \log Pl - \log p \approx \frac{1}{2} s_{1p}^2 + \frac{1}{6}\Sigma w_i^1 p_{1i}^3$$

$$(21) \qquad \log P - \log p \approx -\frac{1}{2} s_{1p}^2 + \frac{1}{6}\Sigma w_i^1 p_{1i}^3 \ .$$

These tell us that log Pl exceeds log p by about half of the variance $s_{1p}^2 = \Sigma w_i^1 p_{1i}^2$ and log p, again, exceeds log P by about the same amount. This explains completely why Pl, p and P are found in different tines of Fisher's fork. These three indices differ from each other and Pl > p > P unless the variance in the price changes is zero when they are equal. In the same way, inserting $c_i = w_i^0$ we get for (1), (4) and (8):

$$(22) \qquad \log L - \log l \approx \frac{1}{2} s_{0p}^2 + \frac{1}{6}\Sigma w_i^0 p_{0i}^3$$

$$(23) \qquad \log Lh - \log l \approx -\frac{1}{2} s_{0p}^2 + \frac{1}{6}\Sigma w_i^0 p_{0i}^3 \ .$$

Thus L > l > Lh, the relative differences being approximately equal to half of the variance of the price changes $s_{0p}^2 = \Sigma w_i^0 p_{0i}^2 \approx s_{1p}^2$. This explains why L, l and Lh are found in different tines of Fisher's fork.

If it so happens - as in the case of Fisher's data - that L and P are approximately equal, then Pl > p > P ≈ L > l > Lh, and the relative differences between any two consecutive indices are approximately equal to half of the variance in the price changes. Furthermore $F = \sqrt{P \cdot L}$, $t = \sqrt{p \cdot l}$ and even[1] $\sqrt{Pl \cdot Lh}$, being means of indices deviating symmetrically from the middle tine, all belong to the middle tine of unbiased index numbers. This is the essence of Fisher's Five-tined Fork.

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1) This is Fisher's formula no. 109, which he classifies in the border line of 'good' and 'very good' index number formulas.

Figure 2. Explanation of Fisher's Five-tined Fork

log Pl	⟶	•	2 +			
log p	⟶	•	1 +			
log P	⟶	•	0	•	⟵	log L
			1 −	•	⟵	log l
			2 −	•	⟵	log Lh

However, Fisher's 'quantum theory' of index numbers is not generally valid because, instead of $P \approx L$, we may have, e.g., $p \approx l$. This happens if the value shares remain approximately constant, $w_i^0 \approx w_i^1$, i.e., the commodities are on the average normally elastic. In this case we have a three-tined fork $Pl \approx L > p \approx l > P \approx Lh$:

Figure 3. A three-tined Fork

log Pl	⟶	•	1 +	•	⟵	log L
log p	⟶	•	0	•	⟵	log l
log P	⟶	•	1 −	•	⟵	log Lh

The upper tine (1+) of this three-tined fork contains Pl and L, while the middle tine now contains p, l and, e.g., F, t and $\sqrt{Pl \cdot Lh}$ as before, the lower tine containing P and Lh. Now p and l are unbiased index numbers while L and P have respectively one dose of upward and downward bias.

Like Fisher we call an index number unbiased in a given situation if it is included in the middle tine of the corresponding fork constructed of the representative two groups of indices of the figures 2 and 3. As is evident from the geometry of the problem the indices of the pairs (Pl, Lh), (p,l) and (P,L) are always located symmetrically with respect to the middle tine and thus their symmetric means, e.g. $\sqrt{Pl \cdot Lh}$, t and F, are always unbiased. Thus an index number formula f is unbiased in a given situation if it is approximately equal to e.g. Fisher's ideal index F, i.e. log (f/F) is only a fraction of variance in the price changes.

Our three tined fork occurs in connection with commodities for which the price and quantity ratios are strongly negatively correlated, so that the value shares remain approximatively constant. This problem was discussed by Fisher (1922) on p. 237-240, 314-317, 410-412 and 428 unsatisfactorily. Fisher tried to show that log l is unbiased only if the negative correlation between the price and quantity ratios p_i^1/p_i^0 and q_i^1/q_i^0 is perfect, Fisher (1922) p. 428: "If the price and quantity elements are thus correlated to the extreme limit of 100 per cent, the downward bias of 23 will be completely abolished. In the present case, where correlation is -88 per cent, the bias is nearly abolished." This analysis is inadequate.

We derive at the end of the paper an exact formula for the logarithmic difference between l and p which solves the problem.

These situations are not the most likely to be met in practice. The situation usually encountered in analyzing, e.g., consumption data would be somewhere between them: neither L nor l is unbiased but L has a small upward and l a small downward bias compared to unbiased index numbers such as F or t. If these biases of L and l are equal in size we have L ≈ p and P ≈ l, which leads to the following new five-tined fork

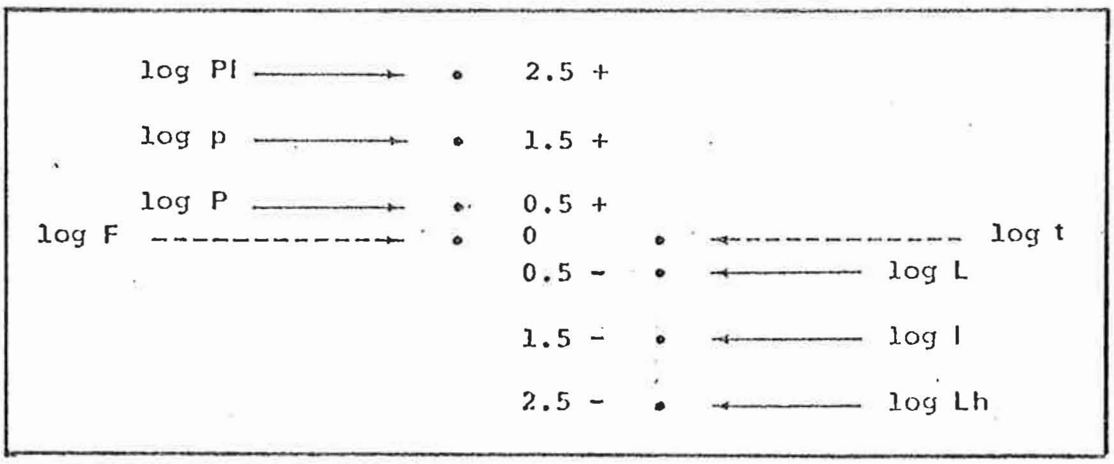
Figure 4. A new five-tined fork

log Pl →	•	1.5 +		
log p →	•	0.5 +	• ←	log L
log F - - - →	•	0	• ← - - -	log t
log P →	•	0.5 −	• ←	log l
		1.5 −	• ←	log Lh

The unbiased index numbers such as $F = \sqrt{P \cdot L}$, $t = \sqrt{l \cdot p}$ and $\sqrt{Pl \cdot Lh}$ shown by dotted arrows are situated half way between $L \approx p$ and $P \approx l$. Thus the biases of the latter are now half the former dose of bias, i.e., of the order of $\frac{1}{4}s^2_p$. Thus the three middle tines of this new fork are closer to each other than in Fisher's fork.

Actually we need not have any of the former cases but the two groups of indices $PL > p > P$ and $L > l > Lh$ may be located quite freely relative to each other. In a situation well explained by the homothetic demand theory we have according to equations (9) and (10) $P_p \leq P(p^1,p^0;\ q^0) = P(p^1,p^0;\ q^1) \leq L_p$ and thus usually $P < L$. We might, e.g., have a seven-tined fork where $\log PL > \log p > \log L > \log F \approx \log t > \log P > \log l > \log Lh$. Here the five middle tines are quite close to each other and only the uppermost and lowermost tines are clearly separated from all the other ones. On the other hand, if the data is well explained by the homothetic production theory we have conversely $P_p \geq P(p^1,p^0;\ q^0) = P(p^1,p^0.\ q^1) \geq L_p$ and thus usually $P > L$. Here we have another seven-tined fork, where the indices disperse more widely:

Figure 5. A seven-tined fork



As a summary we have to recognize that, e.g., Fisher's Ideal Index F and the Törnqvist index t always belong to the middle tine of unbiased index numbers while Pl, P, p, L, l and Lh are all biased up or down in some situations.

3. HOW ARE THE TWO GROUPS OF INDICES LOCATED RELATIVE TO EACH OTHER: A THEORY OF THE 'WEIGHT BIAS'

Next we derive an exact and general expression for the logarithmic difference between p and l, which determines the relative position of the two groups of indices {Pl, p, P} and {L, l, Lh} using respectively new and old value shares as weights. Thus what we are going to give will be essentially a quantitative theory of the 'weight bias'. We have

$$\text{(24)} \qquad \log p - \log l = \Sigma(w_i^1 - w_i^0)\log(p_i^1/p_i^0) .$$

There are many useful approximations to the change in the value share, $\Delta w_i^1 = w_i^1 - w_i^0 = v_i^1/V^1 - v_i^0/V^0$, e.g. Theil (1967) p. 202 extensively uses

$$\text{(25)} \qquad w_i^1 - w_i^0 \approx \tfrac{1}{2}(w_i^1 + w_i^0)\ [\log(v_i^1/v_i^0) - \log(V^1/V^0)]$$

$$= \tfrac{1}{2}(w_i^1 + w_i^0)\ \dot{v}_i .$$

The approximation error is of the third degree in the log-changes $\log(v_i^1/v_i^0)$ and $\log(V^1/V^0)$. This leads to

$$\text{(26)} \qquad \log p - \log l \approx \Sigma\tfrac{1}{2}(w_i^1 + w_i^0)\log(p_i^1/p_i^0)\ \dot{v}_i$$

$$= \Sigma\tfrac{1}{2}(w_i^1 + w_i^0)\dot{p}_i\dot{v}_i$$

$$= \operatorname{cov}(\dot{p}, \dot{v}) \quad \text{, where}$$

$$\text{(27)} \qquad \dot{p}_i = \log(p_i^1/p_i^0) - \Sigma\tfrac{1}{2}(w_i^1 + w_i^0)\log(p_i^1/p_i^0) = \log(p_i^1/p_i^0) - \log t$$

and $\operatorname{cov}(\dot{p}, \dot{v})$ is calculated using the weights $\frac{1}{2}(w_i^1 + w_i^0)$. For the ideas behind such covariances, see Theil (1967) or Rajaoja (1958).

We know, however, the exact equation corresponding to (25):

$$(28)\qquad w_i^1 - w_i^0 = L(w_i^1, w_i^0)\log(w_i^1/w_i^0)$$
$$= L(w_i^1, w_i^0)\,[\log(v_i^1/v_i^0) - \log(V^1/V^0)],$$

where the first line is in fact the definition of the logarithmic mean $L(w_i^1, w_i^0)$, see Vartia (1974, 1976). Thus, identically,

$$(29)\qquad \log p - \log l = \Sigma L(w_i^1, w_i^0)\ \log(p_i^1/p_i^0)\ \dot{v}_i$$
$$= (\Sigma L(w_j^1, w_j^0))\ \Sigma \bar{w}_i \dot{p}_i \dot{v}_i$$
$$= (1-\theta)\,\mathrm{cov}(\dot{p}, \dot{v}) \quad ,$$

where $\bar{w}_i = L(w_i^1, w_i^0)/\Sigma L(w_j^1, w_j^0)$ are the weights of Vartia Index II, (see Vartia (1976) and Sato (1976)) and now $\dot{p}_i = \log(p_i^1/p_i^0) - \Sigma \bar{w}_i \log(p_i^1/p_i^0)$. Because $\theta \geq 0$ is, for small log-changes $\log(w_i^1/w_i^0)$, a very small number

$$(30)\qquad \theta = 1 - \Sigma L(w_j^1, w_j^0)$$
$$\approx \frac{1}{12}\,\Sigma \frac{1}{2}(w_i^1 + w_i^0)\,[\log(w_i^1/w_i^0)]^2$$

we have apart from terms of the third degree in $\log(w_i^1/w_i^0)$

$$(31)\qquad \log p - \log l \approx \mathrm{cov}(\dot{p}, \dot{v}) \ .$$

This formula determines the relative positions of p and l and therefore of the two groups of indices {Pl, p, P} and {L, l, Lh} using respectively new and old value shares as weights. If $w_i^0 = w_i^1$ for all i we have trivially $p = l$. The some happens if the price and value log-changes are uncorrelated or $\mathrm{cov}(\dot{p}, \dot{v}) = 0$. Note that $\log p \gtreqless \log l$ if and only if $\mathrm{cov}(\dot{p}, \dot{v}) \gtreqless 0$, so that $\mathrm{cov}(\dot{p}, \dot{v})$ and a variance in the price changes $s_p^2 = \mathrm{cov}(\dot{p}, \dot{p})$ determine the type of our fork. Knowing only three parameters, $a = \log t$, $b = \mathrm{cov}(\dot{p}, \dot{v})$ and $c = \frac{1}{2}s_p^2$, we may approximately estimate all the indices considered in our paper.

When $\dot{v}_i = \dot{p}_i + \dot{q}_i$ (i.e., the factor reversal test $P_0^1 Q_0^1 = V^1/V^0$ applies to the index number formula used in the calculation of the logarithmic deviations) we have

$$(32) \qquad \log p - \log 1 \approx \text{cov}(\dot{p},\dot{v}) = s_p^2 + \text{cov}(\dot{p},\dot{q}) \quad ,$$

where $s_p^2 = \text{cov}(\dot{p},\dot{p})$ is the variance of the price changes and

$$(33) \qquad \text{cov}(\dot{p},\dot{q}) = \Sigma\bar{w}_i \dot{p}_i \dot{q}_i$$

is the covariance of price and quantity log-changes.[1)] For instance, the logarithmic quantity deviation

$$(34) \qquad \dot{q}_i = \log(q_i^1/q_i^0) - \log Q_0^1$$

$$= \log(q_i^1/q_i^0) - \Sigma\bar{w}_i \log(q_i^1/q_i^0)$$

is positive if the relative change in the quantity of a_i consumed, $\log(q_i^1/q_i^0)$, is greater than the relative change in the quantity of total consumption, $\log Q_0^1$. This means that the quantity of a_i has increased more than the average quantity of consumption. The covariance of price and quantity log-changes (33) is negative if positive (negative) price deviations $\dot{p}_i$ are associated with negative (positive) quantity deviations $\dot{q}_i$, see Theil (1967).

This should be the case according to demand theory (if real consumption does not change much or under homotheticity) because, if the price of a_i increases more than the average prices ($\dot{p}_i>0$), the consumer would decrease his consumption of a_i or at least increase it by less than the average volume of consumption ($\dot{q}_i<0$). Only in the nonhomothetic case, when real consumption changes, may positive deviations of price changes $\dot{p}_i>0$ on the average be associated

1) This is particularly interesting because $\log P - \log L \approx \text{cov}(\dot{p},\dot{q})$ as will be shown later.

with positive deviations of quantity changes $\dot{q}_i > 0$. This may happen if the ceteris paribus effects of price change deviations $\dot{p}_i > 0$ are eliminated by positive income effects - e.g., the commodities for which the prices increase more than average prices happen to be luxuries, which react strongly to rising real income.[1)]

We can write for $\text{cov}(\dot{p},\dot{q})$, as for any covariance,

$$(35) \qquad \text{cov}(\dot{p},\dot{q}) = s_p s_q \, r(\dot{p},\dot{q}) \quad , \text{ where}$$

$$(36) \qquad s_p = \sqrt{s_p^2} = (\Sigma \bar{w}_i \dot{p}_i^2)^{1/2}$$

$$(37) \qquad s_q = \sqrt{s_q^2} = (\Sigma \bar{w}_i \dot{q}_i^2)^{1/2}$$

$$(38) \qquad r(\dot{p},\dot{q}) = \text{cov}(\dot{p},\dot{q})/s_p \cdot s_q \quad \in [-1,1]$$

Here s_p and s_q are the standard deviations of price and quantity log-changes and $r(\dot{p},\dot{q})$ is the correlation between the price and quantity log-changes.

An exact condition for the equality of l, p and $t = \sqrt{l \cdot p}$ according to (29) and (32) may be written

$$(39) \qquad \text{cov}(\dot{p},\dot{v}) = s_p^2 + \text{cov}(\dot{p},\dot{q}) = 0$$

$$(40) \qquad r(\dot{p},\dot{q}) = -(s_p/s_q)$$

If the standard deviations in price and quantity log-changes are equal, $s_p = s_q$, then their negative correlation $r(\dot{p},\dot{q})$ should be -100 % (as Fisher demanded) in order that l (or p) could be 'unbiased'. A much lower

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1) Cf. Theil (1967) p. 254. According to equations (9) and (10) we may have L < P only in the nonhomothetic case and (because logP-logL≈cov($\dot{p}$,$\dot{q}$) only then cov($\dot{p}$,$\dot{q}$) may be definitely positive.

correlation is sufficient if $s_p < s_q$. Note that log L and log P differ now from log t ≈ log F by approximately $\frac{1}{2} s_p^2 \approx -\frac{1}{2} \operatorname{cov}(\dot{p},\dot{q})$ as is shown in our three-tined fork.

If the standard deviation of the quantity changes s_q is much smaller than s_p (as may be the case for necessities with low income and price elasticities) then (40) cannot be satisfied. In this case we have

$$(41) \qquad r(\dot{p},\dot{q}) < -(s_p/s_q)$$

$$(42) \qquad \operatorname{cov}(\dot{p},\dot{v}) = s_p^2 + \operatorname{cov}(\dot{p},\dot{q}) > 0$$

$$(43) \qquad \log p \approx \log 1 + \operatorname{cov}(\dot{p},\dot{v}) > \log 1 \ .$$

It is even possible that $\log p < \log 1$, which happens if

$$(44) \qquad r(\dot{p},\dot{q}) < 0 \text{ and } |r(\dot{p},\dot{q})| > s_p/s_q \ .$$

This implies that $s_p < s_q$. The condition (44) is not probable if the periods from which our data (p_i, q_i) comes are long, say one year. In the analysis of, e.g., monthly data it may well be satisfied because of wild fluctuations in the quantity log-changes.

To sum up:

1. If the variance of the price changes s_p^2 is considerably greater than the variance of the quantity changes s_q^2, then $\log p > \log 1$.

2. If the variance of the price changes s_p^2 is small compared to s_q^2 and the price and quantity changes are negatively correlated, then we may have $\log p < \log 1$.

The relationship between log p and log l makes it possible to derive a useful expression for the difference between log P and log L. Subtracting (22) from (21) and inserting (26) we get

$$(45)\qquad \log P - \log L \approx (\log p - \log l) - \tfrac{1}{2}(s_{1p}^2 + s_{0p}^2)$$

$$\approx \operatorname{cov}(\dot{p},\dot{q}) + s_p^2 - \tfrac{1}{2}(s_{1p}^2 + s_{0p}^2) \; .$$

Consider the variances of the price changes:

$$(46)\qquad \tfrac{1}{2}(s_{1p}^2 + s_{0p}^2) = \Sigma\tfrac{1}{2}(w_i^1 + w_i^0)\,[\log(p_i^1/p_i^0)]^2 - \tfrac{1}{2}(\log p)^2 - \tfrac{1}{2}(\log l)^2$$

$$= \Sigma\tfrac{1}{2}(w_i^1 + w_i^0)\,[\log(p_i^1/p_i^0) - \log t]^2$$

$$+ (\log t)^2 - \tfrac{1}{2}(\log p)^2 - \tfrac{1}{2}(\log l)^2$$

$$= s_p^2 + \tfrac{1}{4}(\log p - \log l)^2$$

Therefore, apart from terms of the third degree in $\dot{p}_i$ and $\dot{q}_i$,

$$(47)\qquad \log P - \log L \approx \operatorname{cov}(\dot{p},\dot{q}) + \tfrac{1}{4}(\operatorname{cov}(\dot{p},\dot{v}))^2$$

$$\approx \operatorname{cov}(\dot{p},\dot{q}) \; .$$

Here the covariance is calculated using the weights $\frac{1}{2}(w_i^1 + w_i^0)$ and the deviations $\dot{p}_i$ and $\dot{q}_i$ are, from the corresponding Törnqvist indices, log t_p and log t_q.

The covariance $\operatorname{cov}(\dot{p},\dot{q})$ may be calculated using the weights $\bar{w}_i$ of any superlative log-change index number (e.g., the weights of Vartia Index II as in (33)) without invalidating (47). Thus the covariance between price and quantity log-changes causes L and P to deviate from each other. This has been qualitatively known[1] but the relationship (47) seems to be new. However, Bortkiewicz (1922, 1924) has derived a very similar and exact relationship for the ratio of P and L, see Allen (1975) p. 63:

$$(48) \qquad P/L = 1 + r\sigma_p\sigma_q/L_qL_p$$

$$= \text{cov}\ (p_i^1/p_i^0,\ q_i^1/q_i^0)/L_qL_p,$$

where r is the coefficient of correlation between price and quantity relatives, σ_p and σ_q are their standard deviations and $\text{cov}(p_i^1/p_i^0,\ q_i^1/q_i^0) = r\sigma_p\sigma_q$ their covariance, all calculated using w_i^0 as the weights. Our formula (47) seems to be more easily combined with other formulas.

4. COMMENTS ON SOME UNBIASED AND 'SUPERLATIVE' INDEX NUMBERS

Our method applies as well to index numbers belonging always to the middle tine of unbiased index numbers.

Calculate, e.g., the difference between

$$(49) \qquad \log t = \tfrac{1}{2}(\log p + \log l) = \Sigma\tfrac{1}{2}(w_i^1+w_i^0)\ \log\ (p_i^1/p_i^0)$$

and

$$(50) \qquad \log F = \tfrac{1}{2}(\log P + \log L)\ .$$

Summing (21) and (22) we get

$$(51) \qquad \log F - \log t \approx -\tfrac{1}{4}(s_{1p}^2 - s_{0p}^2) + \tfrac{1}{12}\ (\Sigma w_i^1 \dot{p}_{1i}^3 + \Sigma w_i^0 \dot{p}_{0i}^3)\ .$$

For the difference between the variances we get by direct calculation

$$(52) \qquad s_{1p}^2 - s_{0p}^2 = \Sigma(w_i^1 - w_i^0)\,[\log\,(p_i^1/p_i^0)\,]^2 - (\log\, p)^2 + (\log\, l)^2$$

$$= \Sigma(w_i^1 - w_i^0)\,[\log\,(p_i^1/p_i^0) - \log\, t]^2 = \Sigma(w_i^1 - w_i^0)\dot{p}_i^2 .$$

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1) See, e.g., Fisher (1922) p. 411, Samuelson & Swamy (1974) p. 592 .

Inserting Theil's approximation (25) into (52) leads to

$$(53) \qquad \log F - \log t \approx -\frac{1}{4}\Sigma\frac{1}{2}(w_i^1 + w_i^0)\dot{p}_i^2\dot{v}_i$$

$$+\frac{1}{6}\Sigma\frac{1}{2}(w_i^1 + w_i^0)\dot{p}_i^3$$

$$\approx -\frac{1}{4}\operatorname{cov}(\dot{p}^2,\dot{v}) + \frac{1}{6}\operatorname{cov}(\dot{p}^2,\dot{p}) \; .$$

This shows that the logarithmic difference between F and t is of the third degree in the deviations of price and value log-changes or very small indeed. In other words, log F and log t are quadratic approximations of each other; quadratic in variables $\dot{p}_i$ and $\dot{v}_i \approx \dot{p}_i + \dot{q}_i$. Note that the variance of the price changes s_p^2 or the covariance $\operatorname{cov}(\dot{p},\dot{q})$ has no effect on their difference log F - log t, which depends only on the 'skewness' or the other third degree properties of the two dimensional distribution of the pair $(\log(p_i^1/p_i^0), \log(q_i^1,q_i^0)$ with the weights $\frac{1}{2}(w_i^1 + w_i^0)$. The covariances in (53) may be combined in a variety of ways to get other approximations.

We conclude that there is no apparent tendency for F to be greater than t, or vice versa, as could have been expected from all of our forks. It is no accident that Fisher (1922) p. 265 places only 14 formulas ahead of t, or his 123.

By similar arguments but starting from (20) and (23) we get

$$(54) \qquad \log \sqrt{Pl \cdot Lh} - \log t \approx \frac{1}{4}\operatorname{cov}(\dot{p}^2,\dot{v}) + \frac{1}{6}\operatorname{cov}(\dot{p}^2,\dot{p}),$$

which shows that even from usually badly biased index numbers we may get a very good formula.

Finally we will show some connections with the economic theory of index numbers. Our moment means $P(\alpha,c) = [\Sigma c_i(p_i^1/p_i^0)^\alpha]^{1/\alpha}$ using the weights $c_i = w_i^0$ are Economic Price Indices corresponding to the generalized CES utility function given, e.g., by Lloyd (1975), if $\alpha = 1 - \sigma$, where σ is the constant elasticity of substitution. We have, e.g., $P(1,w^0) = L$, $P(0, w^0) = 1$ and $P(-1, w^0) = Lh$ for $\sigma = 0$, $\sigma \to 1$ and $\sigma = 2$ respectively, which here are thus 'exact index numbers', to use Diewert's (1976) terminology.

We must have here L = P, l = p and Lh = Pl if $\sigma = 0$, $\sigma \to 1$ and $\sigma = 2$ respectively, because the pairs (L,P), (l,p) and (Lh, Pl) are time antitheses of each other and the Economic Price Index equals in the homothetic case its time antithesis (i.e., satisfies the time reversal test).[1] Therefore Fisher's Five-tined Fork corresponds to the case $\sigma = 0$ (zero substitution case, $q^1 = kq^0$), our three-tined fork to the case $\sigma \to 1$ (or Cobb-Douglas case, $w_i^0 = w_i^1$), while $\sigma = 2$ results in a new five-tined fork, where $\log L > \log l > \log Lh = \log Pl > \log p > \log P$. Here the substitution effects are unusually strong and now L and P have a double bias in respect to the unbiased $Lh = Pl \approx F \approx t$. The assumptions leading to these cases are, however, rather special and the formulas Pl, p, P, L, l and Lh give all in turn biased results as our analysis reveals.

Diewert (1976) defines the quadratic mean of order r unit cost function $c_r(p)$ as follows

$$(55) \qquad c_r(p) = [\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} p_i^{r/2} p_j^{r/2}]^{1/r}, \; a_{ij} = a_{ji}.$$

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1) See Samuelson & Swamy (1974). Curiously the Vartia Index II, an ideal log-change index $\Sigma \bar{w}_i \log(p_i^1/p_i^0)$ using weights $\bar{w}_i$ given in (29), is exact here for all values of σ as proved by Sato (1976).

He shows that the Economic Price Index corresponding to it is the quadratic mean of order r price index P_r:

$$(56) \qquad P_r = [\Sigma w_i^0 (p_i^1/p_i^0)^{r/2} / \Sigma w_i^1 (p_i^1/p_i^0)^{-r/2}]^{1/r},$$

which may be written using geometric means of the moment means as follows

$$(57) \qquad P_r = [P(r/2, w^0)\ P(-r/2,\ w^1)]^{1/2}.$$

We have e.g. $P_2 = F$, $P_0 = t$ and $P_{-2} = \sqrt{Pl \cdot Lh}$, which all belong to the middle tines in our forks. Just as before we get for the relative difference between P_r and t:

$$(58) \qquad \log P_r - \log t \approx -\frac{r}{8}\operatorname{cov}(\dot{p}^2,\ \dot{v}) + \frac{r^2}{24}\operatorname{cov}(\dot{p}^2,\ \dot{p}).$$

Thus P_r and t are for all r and small price and value deviations $\dot{p}_i$ and $\dot{v}_i$ very accurate approximations of each other and therefore P_r is always unbiased, i.e. P_r is contained for all r in the middle tines of our forks.

Inserting (53) into (58) the relative difference between P_r and F is derived. Diewert (1976) shows, using the demand or production theory, that the P_r:s and t are 'superlative index numbers' in a specified sense and are therefore good approximations of each other. Equation (58) expresses approximately the same thing without any assumptions about, e.g., the maximization behaviour of the economic agents. To derive our equation (58) arithmetic alone was needed.

REFERENCES:

ALLEN, R.G.D., (1975): "Index numbers in theory and in practice", The Macmillan Press Ltd, London.

von BORTKIEWICZ, L., (1922, 1924): "Zweck und Struktur einer Preisindexzahl", Nordisk Statistisk Tidskrift, 1 (1922) and 3 (1924).

DIEWERT, W.E., (1976): "Exact and superlative index numbers", Journal of Econometrics, 4, 115-145.

FISHER, F. & SHELL, K., (1972): "The economic theory of price indices", New York.

FISHER, I., (1922): "The making of index numbers", The Riverside Press, Cambridge.

FRISCH, R., (1936): "Annual survey of general economic theory: the problem of index numbers", Econometrica, 4, 1-38.

HARDY, G.H. & LITTLEWOOD, J.E. & POLYA G., (1962): "Inequalities", Cambridge.

LEONTIEF, W., (1936): "Composite commodities and the problem of index numbers", Econometrica, 4, 39-59.

LLOYD,P.J., (1975): "Substitution effects and biases in nonture price indices", The American Economic Review, June 1975.

RAJAOJA, V., (1958): "A study in the theory of demand functions and price indexes", Societas Scientiarum Fennica, Helsinki.

SAMUELSON, P.A. & SWAMY, S., (1974): "Invariant economic index numbers and canonical quality: Survey and synthesis", The American Economic Review, 64, 566-593.

SATO, K., (1976): "The ideal log-change index number", The Review of Economics and Statistics, 58, 223-8.

THEIL, H., (1967): "Economics and information theory", Chicago.

TÖRNQVIST, L., (1936): "Levnads kostnadsindexerna i Finland och Sverige, deras tillförlitlighet och jämförbarket", Ekonomiska Samfundets Tidskrift, 37, 1-35.

VARTIA, Y., (1974): "Relative changes and economic indices", (A Licensiate Thesis in Statistics), University of Helsinki, Department of Statistics, An unpublished paper.

VARTIA, Y., (1976a): "Ideal log-change index numbers", Discussion paper no 2, The Research Institute of the Finnish Economy, forthcoming in the Scandinavian Journal of Statistics.