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INCOMPLETE ELLIPSOIDAL RESTRICTIONS IN LINEAR MODELS*

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by

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Abstract. This paper considers the estimation of parameters of a general linear model under ellipsoidal restrictions. The restrictions are assumed to define an ellipsoid whose dimension is lower than that of the whole parameter space. Such restrictions are called incomplete. It is shown that certain estimators constructed for this situation in the literature have unsatisfactory limiting behaviour when the the ellipsoid either grows infinitely large or shrinks to a point. On the other hand, an estimator obtained by a straightforward generalization of the minimax estimator of Kuks and Olman (1972) does not cause any problems in this respect.

Keywords. Minimax estimation, prior information, quadratic risk, partial minimax estimation, restricted least squares

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1. Introduction

Many authors from Kuks and Olman (1972) and Bunke (1975) to Toutenburg (1982, chapter 4) and Pilz (1986), have considered the estimation of the parameter vector of the general linear model under ellipsoidal restrictions. The theory has been generalized to situations where the dimension of these restrictions is lower than that of the parameter space. Such restrictions are called incomplete for brevity. We shall demonstrate that some of the resulting estimators do not behave as desired when the the ellipsoid either shrinks to a point or inflates to infinity. On the other hand, there is also a well-known estimator making use of incomplete ellipsoid restrictions such that its limiting behaviour is completely reasonable. This estimator may be interpreted as a limiting estimator to the Kuks and Olman (1972) estimator.

We shall consider the estimator of Kuks and Olman in sections 2 and 3. The next two sections are devoted to two other recent approaches to the problem of using incomplete ellipsoidal restrictions. Section 5 offers conclusions.

2. Incomplete restrictions as a limiting case

Consider the linear model

$$y = X\beta + \varepsilon, \quad E\varepsilon = 0, \quad \mathrm{cov}(\varepsilon) = \sigma^2 I \qquad (2.1)$$

where y and ε are stochastic $n \times 1$ vectors and X is a fixed $n \times p$ matrix, $\mathrm{rank}(X) = p$. Assume prior information

$$(\beta - \beta_0)'R'R(\beta - \beta_0) \leq \sigma^2 k^{-1} \qquad (2.2)$$

where R is an $m \times p$ matrix with finite elements and rank(R) = m. Consider the family of estimators

$$\tilde{\beta} = \beta_0 + C'(y - X\beta_0) \; . \qquad (2.3)$$

Set $R(\bar{\beta},A) = E(\bar{\beta} - \beta)'A(\bar{\beta} - \beta)$ where $A \geq 0$. Then

$$R(\tilde{\beta},A) = E(\tilde{\beta}-\beta)'A(\tilde{\beta} - \beta)$$

$$= \sigma^2 \mathrm{tr}ACC' + (\beta - \beta_0)'(C'X - I)'A(C'X - I)(\beta - \beta_0). \qquad (2.4)$$

Following Kuks and Olman (1972), assume A = aa'. Set N = [R' M']' where M is a $(p - m) \times p$ matrix with finite elements and rank(M) = p - m such that N is of full rank p. Approximating the squared bias using the Cauchy-Schwarz inequality yields

$$(\beta - \beta_0)'(C'X - I)aa'(C'X - I)(\beta - \beta_0)$$

$$= [a'(C'X - I)N^{-1}N(\beta - \beta_0)]^2$$

$$\leq (\beta - \beta_0)'(R'R + M'M)(\beta - \beta_0)a'Ha$$

$$\leq \{\sigma^2 k^{-1} + (\beta - \beta_0)'M'M(\beta - \beta_0)\} a'Ha \qquad (2.5)$$

where

$$H = (C'X - I)'(N'N)^{-1}(C'X - I) \; .$$

Note that the right-hand side of (2.5) is unbounded in $\mathcal{B} = \{\beta: (\beta - \beta_0)'R'R(\beta - \beta_0) \leq \sigma^2 k^{-1}\}$. A sufficient condition for boundedness is

$$(\beta - \beta_0)'M'M(\beta - \beta_0) \leq \sigma^2(\lambda^2 k)^{-1} , \quad \lambda > 0 . \tag{2.6}$$

Inequality (2.6) implies that we in fact have

$$(\beta - \beta_0)'N'DN(\beta - \beta_0) \leq \sigma^2 k^{-1} \tag{2.7}$$

where $D = \text{diag}(I_m, \lambda^2 I_{p-m})$. In that case, setting $C' = (X'X + kN'DN)^{-1}X'$ in (2.3) we have the customary minimax estimator

$$\begin{aligned} b_N(k) &= \beta_0 + (X'X + kN'DN)^{-1}X'(y - X\beta_0) \\ &= (X'X + kR'R + \lambda^2 kM'M)^{-1}(X'y + kR'r_0 + \lambda^2 kM'm_0) \end{aligned} \tag{2.8}$$

where $r_0 = R\beta_0$ and $m_0 = M\beta_0$; cf. Kuks and Olman (1972).

When λ decreases, the size of the ellipsoid (2.6) increases. The prior information conveyed by (2.6) then becomes more and more vague and relatively less useful in estimation. As $\lambda \to 0$, it is reasonable to expect (2.8) to converge in probability towards an estimator which only makes use of the m-dimensional restriction (2.2). Indeed,

$$b_R(k) = \underset{\lambda \to 0}{\text{plim}}\, b_N(k) = Z(X'y + kR'r_0) \tag{2.9}$$

where $Z = (X'X + kR'R)^{-1}$. Estimator (2.9) is minimax for all $\lambda > 0$ if (2.6) holds but (2.9) no longer has this property if $\lambda = 0$. Nonetheless, we would like to stress two things. First, when $\lambda = 0$, (2.9) is unbiased and has thus bounded risk if $R\beta = r_0$. Second, as a limit of (2.8) as $\lambda \to 0$, (2.9) is a natural candidate to be selected among members of (2.3) when (2.2) holds.

It is also instructive to see what happens when $\lambda \to \infty$ in (2.8). After some manipulation we obtain

$$\tilde{b}_M(k) = \lim_{\lambda \to \infty} b_N(k) = b_R(k) - ZM'(MZ\,M')^{-1}(Mb_R(k) - m_0). \qquad (2.10)$$

From the assumption (2.6) it now follows that $M\beta = m_0$, and the maximum risk of (2.10) is thus bounded. Estimator (2.10) is a generalization of the heuristic combined estimator of Toutenburg (1980). The argument leading to (2.10) gives it a theoretical motivation. The author assumes that R'R is non-singular. We conclude that this is not necessary for the minimax property if rank(N) = p and $M\beta = m_0$.

3. Special cases

A question of considerable interest is the limiting behaviour of (2.9) when the incomplete ellipsoid either increases beyond any bound ($k \to 0$) or shrinks to a point ($k \to \infty$). In the former case, the vagueness of the prior information increases; in the latter, the information becomes exact and the corresponding restrictions thus linear. Hence, when $k \to 0$, the estimator based on incomplete ellipsoidal restrictions should converge in probability to the least squares estimator; it is easily seen that (2.9) satisfies this property.

Likewise, as $k \to \infty$, (2.9) converges in probability to the restricted least squares estimator

$$b_R = b + UR'(RUR')^{-1}(r_0 - Rb) . \qquad (3.1)$$

The limiting behaviour of (2.9) is thus fully satisfactory.

4. Kozák's estimator

Next we shall extend the above considerations to other estimators of β in (2.1) under (2.2). Kozák (1985) has recently suggested (2.3) with

$$C' = R^- \{(R^-)'(X'X + kR'R)R^-\}(R^-)'X' \qquad (4.1)$$

where R^- is a generalized inverse of R. It is obvious from (2.5) that this estimator has unbounded maximum risk in $\mathcal{B}$. A special case of (4.1) is obtained by defining $R^- = SR'(RSR')^{-1}$ where S is an arbitrary $p \times p$ positive definite matrix. Thus we have

$$b_R^*(k,S) = \beta_0 + SR'(RSR')^{-1}\{(RSR')^{-1}RSZ^{-1}SR'(RSR')^{-1}\}^{-1}$$

$$\times (RSR')^{-1}RSX'(y - X\beta_0)$$

$$= \beta_0 + SR'(RSZ^{-1}SR')^{-1}RSX'(y - X\beta_0) \qquad (4.2)$$

cf. Kozák (1985).

As mentioned above, letting $k \to 0$ in (4.2) is equivalent to relaxing the ellipsoidal constraint (2.2) so that it ultimately vanishes (is satisfied for any $\beta \in R$). However, contrary to our expectations

$$\operatorname*{plim}_{k \to 0} b_R^*(k,S) = \beta_0 + SR'(RSX'XSR')^{-1}RSX'(y - X\beta_0) \neq b.$$

Consider now the case $k \to \infty$ in which (4.2) should preferably converge in probability to (3.1). From (4.2) it follows that

$$b_R^*(\infty , S) = \operatorname*{plim}_{k \to \infty} b_R^*(k,S) = \beta_0. \tag{4.3}$$

Note that (4.3) does satisfy

$$Rb_R^*(\infty , S) = R\beta_0 = r_0.$$

Yet, unlike in (2.9), nothing remains to be estimated when the ellipsoidal restriction (2.2) shrinks to a linear one. Thus the limiting behaviour of (4.2) (or (4.1)) is not satisfactory compared to that of (2.9).

5. Partial minimax estimation

Hering et al. (1984) have introduced another approach to estimating β in (2.1) under (2.2). Their starting-point is the claim that the minimax estimator is applicable only if the whole parameter vector is restricted into an ellipsoid. The authors restrict themselves to a special case of (2.2) where $R = [0, T_2^{1/2}]$ and $T_2^{1/2}$ is a symmetric positive definite $m \times m$ matrix. Let $\beta = (\beta_1', \beta_2')'$ and $\beta_0 = (0', \beta_{02}')'$ be the conformable partitions of β and β_0, respectively, and define $T = \mathrm{diag}(I_{p-m}, T_2)$. The restrictions are thus

$$(\beta_2 - \beta_{02})'T_2(\beta_2 - \beta_{02}) \leq \sigma^2 k^{-1} . \qquad (5.1)$$

Assuming the quadratic risk function (2.4) and $A = aa'$ as before, the partial minimax estimator is defined as (2.3) with

$$C' = H(X'X + \delta T)^{-1}X' + (I - K)Z' . \qquad (5.2)$$

In (5.2), in our notation, $\delta = (k^{-1} + \beta_1'\beta_1)^{-1}$, $K = aa'/a'a$ and Z is an arbitrary $n \times p$ matrix. Without further comment, Hering et al. (1984) limit their attention to the principal solution

$$C' = (X'X + \delta T)^{-1}X'$$

so that

$$\hat{\beta}_{PM}(\delta) = \beta_0 + (X'X + \delta T)^{-1}X'(y - X\beta_0). \qquad (5.3)$$

Estimator (5.3) behaves correctly as $k \to 0$: it converges in probability towards the least squares estimator b. As $k \to \infty$, the restrictions on β_2 become exact (linear), i.e., $\beta_2 = \beta_{02}$. Then, however, (5.3) approaches

$$\hat{\beta}_{PM}((\beta_1'\beta_1)^{-1}) = (X'X + (\beta_1'\beta_1)^{-1}T)^{-1}(X'y + (\beta_1'\beta_1)^{-1}T\beta_0) \qquad (5.4)$$

which does not satisfy the linear restrictions. Operational, at the same time non-linear, variants of (5.4) share the same property.

6. Combining linear and ellipsoidal restrictions

It has been stressed above that a reasonable requirement for an estimator making use of ellipsoidal restrictions is that it approaches an ordinary least squares estimator as these restrictions vanish, i.e., $k \to 0$ in (2.2). In the same vein, any estimator combining linear and ellipsoidal restrictions should converge in probability to a restricted least squares estimator as the ellipsoid expands beyond any bound, i.e., $k \to 0$. This holds in fact for (2.10) and therefore also for the slightly less general estimator of Toutenburg (1980).

As can be seen, we have concentrated on limiting cases of certain alternative estimators, whereas their risk properties have not been a topic for thorough discussion. Comparing estimators in this respect is possible using results from Teräsvirta (1986). However, we have only wanted to demonstrate that (2.9) has a natural interpretation and acceptable limiting behaviour which make it a serious alternative when estimating β under (2.2).

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