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IDEAL LOG-CHANGE

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IDEAL LOG-CHANGE INDEX NUMBERS

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Abstract. Two new 'ideal index number formulas' based on logarithmic changes are presented. Both of them satisfy the time and the factor reversal tests, react qualitatively correctly to extreme price and volume changes but only one of them is consistent in aggregation. The indices are derived using a new representation for the log-change: it is a relative change with respect to the 'logarithmic mean'. The role of the proportionality test is discussed and a weaker form of it is proposed. The article elaborates the problem discussed by Theil (1973) and Sato (1974).

Key words: Ideal index number, factor reversal test, proportionality test, log-change, consistency in aggregation.

1. Introduction

Index numbers based on relative changes rather than on price or volume ratios have recently been given considerable attention, see Christensen and Jorgenson (1970), Merilees (1971), Hulten (1973), Theil (1973) and (1974), Sato (1974).

We want to elaborate and comment upon particularly the papers of Theil and Sato. Let $a_1, a_2, \ldots, a_n$ be the commodities (or the commodity groups) for which the indices will be defined. Denote the value of a_i by v_i (in money units), its quantity by q_i (in physical units), price by $p_i = v_i/q_i$ and value share by $w_i = v_i/\Sigma v_j$. Periods (or places) between which the comparisons are made will be denoted by subscripts 0 and 1 and the price and volume indices as formal ratios P_1/P_0 and Q_1/Q_0.

The logarithmic form of a price index is

$$\log\left(\frac{P_1}{P_0}\right) = \sum_{i=1}^{n} \bar{w}_i \log\left(\frac{p_{1i}}{p_{0i}}\right) \text{, where} \tag{1}$$

the weights $\bar{w}_i$ are 'value shares' of some kind. Walsh (1901) Törnqvist (1936) and Theil (1937) respectively have proposed the following weights:

$$\bar{w}_i = \frac{\sqrt{v_{1i}v_{0i}}}{\Sigma\sqrt{v_{1j}v_{0j}}} = \frac{G(v_{1i},v_{0i})}{\Sigma G(v_{1j},v_{0j})} = \frac{G(w_{1i},w_{0i})}{\Sigma G(w_{1j},w_{0j})} \tag{2}$$

$$\bar{w}_i = \frac{1}{2}\left(\frac{v_{1i}}{\Sigma v_{1j}} + \frac{v_{0i}}{\Sigma v_{0j}}\right) = \frac{1}{2}(w_{1i}+w_{0i}) = M(w_{1i},w_{0i}) \tag{3}$$

$$\bar{w}_i = \frac{\sqrt[3]{\frac{1}{2}(w_{1i}+w_{0i})w_{1i}w_{0i}}}{\Sigma\sqrt[3]{\frac{1}{2}(w_{1j}+w_{0j})w_{1j}w_{0j}}} = \frac{T(w_{1i},w_{0i})}{\Sigma T(w_{1j},w_{0j})} \text{,} \tag{4}$$

where $M(x,y)$, $G(x,y)$ and $T(x,y)$ are arithmetic, geometric and 'Theil' averages respectively. All the weights (2)-(4) are normed to add up to unity, which complicates the expressions (2) and (4). Törnqvist's weights automatically add up to unity. Walsh's weights are proportional to the geometric, Törnqvist's weights to the arithmetic and Theil's weights to the 'Theil' averages of the new and old value shares.

Although the geometric average P_1/P_0 in (1) is in principle the right type of average to be used in index formulas, it has some drawbacks. It has been difficult to define the weights $\bar{w}_i$ in (1) so that the product of the price index and its corresponding volume index Q_1/Q_0 in

$$\log(\frac{Q_1}{Q_0}) = \sum_{i=1}^{n} \bar{w}_i \log(\frac{q_{1i}}{q_{0i}}) \tag{5}$$

is the value ratio $\Sigma v_{1i}/\Sigma v_{0i}$. None of the index formulas defined by the above weights satisfies the factor reversal test, but Theil's formula is constructed so as to give a good approximation. Sato (1974) gives other approximations, for discussion see Theil (1974).

Another drawback is the possible wrong behaviour of (1) when some of the p_{1i} approaches zero, i.e. when a_i becomes a free good. The effect of an individual price change or its contribution to the log-change of the price index is

$$\bar{w}_i \log(\frac{p_{1i}}{p_{0i}}) \; . \tag{6}$$

For Walsh's weights (2) this approaches zero, because the weight decreases faster than the logarithm increases. The same happens (in the limit) when Theil's weights (4) are used. Thus for these two weights the index P_1/P_0 remains unchanged if, ceteris paribus, any of the prices p_{1i} is set equal to zero. Hence, an extreme price reduction of any one commodity does not lower the price level at all!

Theil comments upon this drawback of 'no index changes' of his formula only by stating that there is "no problem of infinite index changes".

By contrast, Törnqvist's weights $\frac{1}{2}(w_{1i}+w_{0i})$ approach a nonzero value $\frac{1}{2}w_{0i}$ when p_{1i} approaches zero, and thus the contribution (6) will be minus infinity. This means that, according to Törnqvist's formula, the price index P_1/P_0 approaches zero together with any of the individual price ratios.

2. Two new ideal log-change index number formulas

We introduce the following refinements of the weights, of which the first was discovered before the publication of Theil's article and the second was constructed to fulfill the requirements set forth by Theil:

$$\bar{w}_i = \frac{L(v_{1i}, v_{0i})}{L(\Sigma v_{1j}, \Sigma v_{0j})} \tag{7}$$

$$\bar{w}_i = \frac{L(w_{1i}, w_{0i})}{\Sigma L(w_{1j}, w_{0j})}, \tag{8}$$

where L(x,y) denotes the 'logarithmic average' defined for positive x and y by

$$\begin{aligned} L(x,y) &= \frac{y - x}{\log y - \log x}, \quad \text{for } x \neq y \\ &= x \quad\quad\quad\quad\;\, , \quad \text{for } x = y. \end{aligned} \tag{9}$$

The corresponding indices defined by the weights (7) and (8) will be called Vartia Indices I and II for easy reference. Both indices where presented in Vartia (1974). Sato (1975) documents an independent rediscovery of (8).

The logarithmic average satisfies the equation

$$\log\left(\frac{y}{x}\right) = \frac{y-x}{L(x,y)} , \tag{10}$$

expressing that the log-change is literally a relative change with respect to the logarithmic average. Using the mean value theorem we get $\min(x,y) \leq L(x,y) \leq \max(x,y)$ for any positive x and y. Furthermore $L(ax,ay) = aL(x,y)$, so that $L(x,y)$ may be considered as an average. We also have

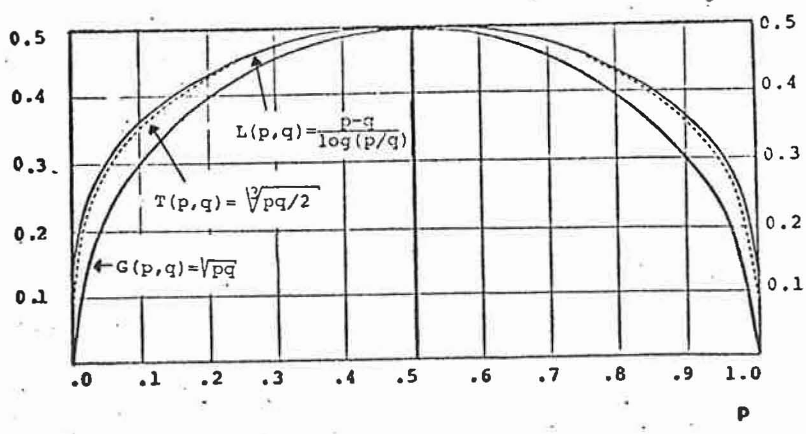
$$G(x,y) \leq T(x,y) \leq L(x,y) \leq M(x,y) , \tag{11}$$

with the equality signs only if $x=y$. For a graphical comparison of these averages we write

$$K(x,y) = (x+y)\ K\left(\frac{x}{x+y} , \frac{y}{x+y}\right) = 2M(x,y)\ K(p,q) , \tag{12}$$

where $K(x,y)$ is any of the averages , $p = x/(x+y)$ and $q = 1-p$. Thus $K(p,q) = \frac{1}{2}K(x,y)/M(x,y)$ indicates the behaviour of the average $K(x,y)$ in compact form. The inequalities (11), for example, are apparent from the following figure. The averages differ considerably only for small values of p and q. A remarkable conclusion is that $T(x,y)$ and $L(x,y)$ are accurate approximations of each other.

Figure: Some averages $K(p,q)$



It is instructive to note that the 'shoulders' of $M(p,q) \equiv 1/2$ for $p = 0$ or $q = 0$ give rise to the 'problem of infinite index changes'. On the other hand 'the raglan-type shoulders' of $G(p,q)$ and $T(p,q)$, which descend too quickly when $p \to 0$ or $q \to 0$, cause the 'problem of no index changes'. In the following we shall prove that the 'shoulders' of $L(p,q)$ are of the right type.

First we prove that our new indices exactly satisfy the factor reversal test. For the weights (7) we have

$$\log\left(\frac{\Sigma v_{1i}}{\Sigma v_{0i}}\right) = \frac{\Sigma v_{1i} - \Sigma v_{0i}}{L(\Sigma v_{1j}, \Sigma v_{0j})} = \Sigma \frac{L(v_{1i}, v_{0i})}{L(\Sigma v_{1j}, \Sigma v_{0j})} \log\left(\frac{v_{1i}}{v_{0i}}\right) \quad (13)$$

$$= \Sigma \bar{w}_i \log\left(\frac{p_{1i}}{p_{0i}}\right) + \Sigma \bar{w}_i \log\left(\frac{q_{1i}}{q_{0i}}\right),$$

which proves the factor reversal property of Vartia Index I. The gist in the proof is the representation (10).

Theil (1973) proved that a sufficient condition for an index of type (1) to satisfy the factor reversal test is

$$\Sigma\bar{w}_i\log(\frac{w_{1i}}{w_{0i}}) = 0 \text{ and } \Sigma\bar{w}_i = 1. \tag{14}$$

Theil constructed his weights (4) so as to fulfil this condition more accurately than Walsh's and Törnqvist's weights. The following shows that the weights (8) satisfy the condition (14) identically, and, thus, the Vartia Index II is just the index Theil and Sato were trying to construct and which was later independently discovered by Sato (1975). For the weights (8) we have

$$\Sigma\bar{w}_i\log(\frac{w_{1i}}{w_{0i}}) = \Sigma\frac{L(w_{1i},w_{0i})}{\Sigma L(w_{1j},w_{0j})}(\frac{w_{1i}-w_{0i}}{L(w_{1i},w_{0i})}) \tag{15}$$

$$= \frac{\Sigma(w_{1i}-w_{0i})}{\Sigma L(w_{1j},w_{0j})} = 0$$

because $\Sigma w_{1i} = \Sigma w_{0i} = 1$. Theil's condition (14) is not necessary, the weights (7) providing a counterexample. Another counterexample is provided by the weights $\bar{w}_i$ of the Fisher's Ideal Index, when it is written in logarithmic form (1). For both of these weights $\Sigma\bar{w}_i \leq 1$ and inequality applies usually.

3. Extreme price changes

To show the correct qualitative behaviour of the new indices when some of the $p_{1i} \to 0$ we consider the terms (17) relating to the commodity a_i of the fundamental decomposition

$$\log(\frac{\Sigma v_{1i}}{\Sigma v_{0i}}) = \Sigma\bar{w}_i\log(\frac{v_{1i}}{v_{0i}}) = \Sigma\bar{w}_i\log(\frac{p_{1i}}{p_{0i}}) + \Sigma\bar{w}_i\log(\frac{q_{1i}}{q_{0i}}) \tag{16}$$

$$\bar{w}_i\log(\frac{v_{1i}}{v_{0i}}) = \bar{w}_i\log(\frac{p_{1i}}{p_{0i}}) + \bar{w}_i\log(\frac{q_{1i}}{q_{0i}}) . \tag{17}$$

The identity (17) gives the contributions of value, price and quantity changes of the commodity a_i to the corresponding logarithmic changes of the aggregates.

We suppose that v_{0j}, p_{0j}, q_{0j}, $j = 1,2,\ldots, n$ are given constants and calculate the consequences of decreasing p_{1i}. For the sake of simplicity we first assume that all the quantities q_{1j} and prices p_{1j}, $j \neq i$ are fixed as well, so that the only changing elements in (17) are p_{1i}, $v_{1i} = p_{1i}\, q_{1i}$ and the weights. The analysis is thus concerned with a variety of hypothetical situations.

We consider the Vartia Index I, which shows the general principle. Since

$$\bar{w}_i = \frac{L(v_{1i}, v_{0i})}{L(\Sigma v_{1j}, \Sigma v_{0j})} \to 0, \text{ when } p_{1i} \to 0 \text{ and } v_{1i} \to 0 \tag{18}$$

the quantity term of (17) vanishes and the value and price terms are equal in the limit. Hence

$$\bar{w}_i \log\left(\frac{p_{1i}}{p_{0i}}\right) \to \bar{w}_i \log\left(\frac{v_{1i}}{v_{0i}}\right) = \frac{v_{1i} - v_{0i}}{L(\Sigma v_{1j}, \Sigma v_{0j})}, \text{ when } p_{1i} \to 0. \tag{19}$$

The last term approaches $-v_{0i}/L(\Sigma v_{1j}, \Sigma v_{0j})$, so that the contribution of an extreme price reduction of a_i to $\log(P_1/P_0)$ is approximately equal to $-w_{0i}$, if $\Sigma v_{1j} \approx \Sigma v_{0j}$. The conclusion holds without supposing that q_{1i} is constant as long as it remains bounded. The same argument may be applied instead of price to volume, so that Vartia Indices react qualitatively correctly to appearing and disappearing commodities. In order to investigate all the effects of the price reduction of any given commodity we should assume a complete set of demand functions.

4. Weight sums and the Proportionality Test

The dissimilarities between the two new indices have been examined in Vartia (1974). Some points deserve to be mentioned here. The weights (7) do not generally add up to unity but their sum is one at the most. At first sight this may seem quite curious because the weights in (1) are usually normed as in e.g. (2)-(4). This is not necessary, however, which is seen by writing Laspeyres' price index in logarithmic form:

$$\log\left(\frac{\Sigma p_{1i}q_{0i}}{\Sigma p_{0j}q_{0j}}\right) = \frac{\Sigma p_{1i}q_{0i} - \Sigma p_{0j}q_{0j}}{L(\Sigma p_{1j}q_{0j}, \Sigma p_{0j}q_{0j})} \tag{20}$$

$$= \sum_i \frac{q_{0j}L(p_{1j}, p_{0j})}{L(\Sigma p_{1j}q_{0j}, \Sigma p_{0j}q_{0j})} \frac{(p_{1i}-p_{0i})}{L(p_{1i}, p_{0i})} = \sum_i \bar{w}_i \log\left(\frac{p_{1i}}{p_{0i}}\right),$$

$$\text{where } \bar{w}_i = \frac{L(p_{1i}q_{0i}, p_{0i}q_{0i})}{L(\Sigma p_{1j}q_{0j}, \Sigma p_{0j}q_{0j})} = w_{0i}\frac{L(p_{1i}/p_{0i}, 1)}{L(\Sigma w_{0j}(p_{1j}/p_{0j}), 1)}.$$

The weights $\bar{w}_i$ in (20) satisfy only $\Sigma\bar{w}_i \leq 1$ as is proved in Vartia (1974). In the same way the weights of the logarithmic forms of Paasche's and Fisher's indices may be deduced and the sums of these weights equal one at the most, being one only under special conditions. From this property of Fisher's Ideal Index we may conclude that no "downward bias" is generally introduced by weights $\bar{w}_i$, whose sum is one at the most.

On the contrary weights (2)-(4) and (8) the sum of which is forced to unity might be claimed to introduce some "upward bias" in the log-change of the index compared to the log-change of the Fisher's Ideal Index. No such bias is, however, possible for the weights (7) or (8) because Vartia Indices I and II satisfy the factor reversal test.

Suppose that the log-changes of the aggregates in (16) are all positive, then it is impossible that e.g. Vartia Index I would have "downward bias" both in the price and volume index log-changes because their sum is the log-change of the value index.

To examine the peculiarities of Vartia Index I we now turn to the so-called Proportionality Test: If all individual prices (quantities) change in the same proportion from 0 to 1, the price (quantity) index should be equal to the common factor of proportionality. Or in symbols: $(p_1 = kp_0 \Rightarrow P(p_1, p_0, q_1, q_0) = k)$ & $(q_1 = kq_0 \Rightarrow Q(q_1, q_0, p_1, p_0) = k)$. We easily find that Vartia Index II satisfies the Proportionality Test always, whereas Vartia Index I satisfies it only if $w_{1i} = w_{0i}$ for all i. The same drawback is shared e.g. by the "superlative" index number formulas of the geometric type presented by Fisher (1922). These may be called "ideal log-change index numbers" because they satisfy both the time and the factor reversal test. Fisher regarded this non-proportionality only as a minor drawback, see Fisher (1922) p. 421. But the problem seems to be more profound.

Let us turn to the economic theory of index numbers and denote, adopting the definitions and the notation of Samuelson and Swamy (1975), the economic price and quantity indices by $p(P^1,P^0;\ Q^\alpha)$ and $q(Q^1,Q^0;\ P^\alpha)$, where Q^α and P^α are the reference quantity and price vectors needed in the definitions. In the general nonhomothetic case we have $p(kP^1,P^0;\ Q^\alpha)=kp(P^1,P^0;\ Q^\alpha)$. Thus the economic price index satisfies even a stronger proportionality test. This stronger proportionality test is not satisfied by the best index number formulas, say, Fisher's formula or Vartia Index II without special assumptions on quantities. We consider here the general nonhomothetic case. In the homothetic case realized prices and quantities always change proportionally at the same time and e.g. the Vartia Index I satisfies then the Proportionality Test.

The economic quantity index, however, does not generally satisfy the Proportionality Test, but usually we have $q(kQ^0,Q^0;\ P^\alpha) \neq k$, see Samuelson and Swamy (1974) pp 576 and 585.

We must conclude that if an index number formula satisfies the factor reversal test, either the price or the volume index behaves in a wrong way (according to the economic theory) if the prices or the quantities change proportionally. Therefore Fisher's formula and the Vartia Index II show the right qualitative behavior as price indices but behave badly as quantity indices. And conversely: The Vartia Index I and e.g. Fisher's superlative geometric indices (Fisher's formulas nos. 323, 325, 1323 and 5323) do not satisfy the Proportionality Test but may give a high order approximation to the economic quantity index.

We could propose to apply Vartia Index II only as a price index and Vartia Index I only as a quantity index, but this would merely hide the problem in their corresponding quantity and price indices defined implicitely by the strong factor reversal test.

Thus any "ideal index number formula" can be criticized from the viewpoint of economic theory of index numbers. This criticism is therefore not directed at the formula but at the factor reversal test, which is in fact an old story. From the descriptive point of view, however, we want to maintain the factor reversal test and therefore a weaker proportionality test is proposed: $p_1=kp_0$ & $q_1=mq_0 \Rightarrow P(p_1,p_0,q_1,q_0)=k$ & $Q(q_1,q_0,p_1,p_0)=m$. This is a symmetric version of the proportionality criterion which is satisfied by both economic price and quantity indices. In this weaker proportionality test prices and quantities are supposed to change proportionally at the same time and in such a situation value shares remain constant.

5. About consistency in aggregation

The quantitative difference between Vartia Index I and e.g. Törnqvist's Index (3), Fisher's Ideal Index and Vartia Index II is for small price and volume changes of the third order of smallness in price and volume changes. Diewert (1975) uses this to prove that his 'superlative indices' are approximately consistent in aggregation

But what is the qualitative difference between the Vartia Indices I and II ? What is the cost of forcing the sum of the weights to unity and thus preserving the interpretation of (1) as a weighted average? The answer seems to be: the Vartia Index I is consistent in aggregation, whereas the Vartia Index II is not.

Consistency in aggregation refers to a situation where the set of commodities $A = \{a_1, a_2, \ldots, a_n\}$ is considered as a union of disjoint subsets A_k, $A = \bigcup_{k=1}^{K} A_k$, and price and volume indices for A are calculated via indices for A_k, $k = 1,2,\ldots, K$.

One way of computing e.g. a price index for A is to calculate it in two stages. First the price indices (and possibly the volume indices) are calculated for every subset A_k of the partition. Thereafter the total index for A is calculated, using the subindices and the same index formula, from the aggregate data for the subsets. An index formula is <u>consistent in aggregation</u> if the value of the index as calculated in two stages necessarily coincides with the value of the index as calculated in an ordinary way.

For instance Laspeyres' formula

$$(P_1/P_0) = \frac{\Sigma p_{1i}q_{0i}}{\Sigma p_{0j}q_{0j}} = \sum_{i=1}^{n} w_{0i}\left(\frac{p_{1i}}{p_{0i}}\right) \tag{21}$$

is consistent in aggregation:

$$\sum_{i=1}^{n} w_{0i}\left(\frac{p_{1i}}{p_{0i}}\right) = \sum_{k=1}^{K}\sum_{A_k} w_{0i}\left(\frac{p_{1i}}{p_{0i}}\right) \tag{22}$$

$$= \sum_{k=1}^{K} W_{0k}\left[\sum_{A_k} w_{0i}(k)\left(\frac{p_{1i}}{p_{0i}}\right)\right] \text{, where}$$

$\sum\limits_{A_k} (\ )_i$ denotes the sum of terms $(\ )_i$ for which $a_i \in A_k$ and

$$w_{0i}(k) = v_{0i}/\sum_{A_k} v_{0j} = \text{value share of } a_j \text{ in } A_k \text{ at } t_0, \tag{23}$$

$$W_{0k} = \sum_{A_k} v_{0j}/\sum_{A} v_{0j} = \text{value share of } A_k \text{ in } A \text{ at } t_0. \tag{24}$$

We may write a similar decomposition for Törnqvist's formula:

$$\sum_{i=1}^{n} \frac{1}{2}(w_{1i}+w_{0i})\log(\frac{p_{1i}}{p_{0i}}) = \sum_{k=1}^{K} \sum_{A_k} \frac{1}{2}(w_{1i}+w_{0i})\log(\frac{p_{1i}}{p_{0i}}) \tag{25}$$

$$= \sum_{k=1}^{K} \frac{1}{2}(W_{1k}+W_{0k})[\sum_{A_k} w_i \log(\frac{p_{1i}}{p_{0i}})]$$

where $w_i = (w_{1i}+w_{0i})/(W_{1k}+W_{0k})$. Unfortunately these weights are not equal to the "conditional weights" $\frac{1}{2}(w_{1i}(k)+w_{0i}(k))$ and thus the expression in brackets is not the value of Törnqvist's index for A_k. In fact the weights w_i of $a_i \in A_k$ cannot be calculated using the data for A_k only, but they depend on total values $\sum\limits_{A} v_{0j}$ and $\sum\limits_{A} v_{1j}$ in a complicated manner.

This shows that Törnqvist's formula is <u>not</u> consistent in aggregation.

Theil (1973) uses a similar decomposition in order to prove the consistency in aggregation of his index formula, but his proof fails for the same reason. Thus, contrary to Theil's proposition, his formula is not consistent in aggregation.

Similar faulty propositions, originating from merely verbal definitions of consistency in aggregation, may be found e.g. in Christensen and Jorgenson (1970) p. 26. A mathematical treatment of this problem is given in Vartia (1974).

Just like Törnqvist's and Theil's formulas the Vartia Index II is not consistent in aggregation. This is, in a sense, the price paid for the weights' adding up to unity.

On the other hand, the Vartia Index I is consistent in aggregation, for we have:

$$\sum_{i=1}^{n} \frac{L(v_{1i},v_{0i})}{L(\sum_A v_{1j},\sum_A v_{0j})} \log(\frac{p_{1i}}{p_{0i}}) = \sum_{k=1}^{K} \sum_{A_k} \frac{L(v_{1i},v_{0i})}{L(\sum_A v_{1j},\sum_A v_{0j})} \log(\frac{p_{1i}}{p_{0i}}) \quad (26)$$

$$= \sum_{k=1}^{K} \tilde{W}_k [\sum_{A_k} \frac{L(v_{1i},v_{0i})}{L(\sum_{A_k} v_{1j},\sum_{A_k} v_{0j})} \log(\frac{p_{1i}}{p_{0i}})].$$

The bracketed expression is the Vartia Index I for A_k, and these indices are aggregated by the same formula, because

$$\tilde{W}_k = L(\sum_{A_k} v_{1j},\sum_{A_k} v_{0j})/L(\sum_A v_{1j},\sum_A v_{0j}). \quad (27)$$

Thus the Vartia Index I is consistent in aggregation.

6. Conclusion

It is proved that Vartia Indices I and II satisfy the time and the factor reversal tests and react qualitatively correctly to extreme price and volume changes, but only the Vartia Index I is consistent in aggregation.

The only known index sharing these properties of Vartia Index I is an exceptional but ingenious kind of index "stumbled on" by Stuvel (1957). Stuvel's index is based on a decomposition of an arithmetic change of value into symmetric price and volume components, while our decomposition is based on logarithmic changes.

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