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EFFICIENT METHODS OF MEASURING WELFARE CHANGE AND COMPENSATED INCOME IN TERMS OF MARKET DEMAND FUNCTIONS

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EFFICIENT METHODS OF MEASURING WELFARE CHANGE AND COMPENSATED INCOME IN TERMS OF MARKET DEMAND FUNCTIONS

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Abstract

When evaluating the effects of economic policy upon individuals the concepts of utility, well-being or welfare change are often used. These are usually unobservables which are approximated using index number calculations or consumer surplus measures based on observable information.

We consider a situation where a consumer maximizes a (unobserved, ordinal) utility function $u(q)$ under the budget restriction. Consumer's market demand system $h(p,C)$ is supposed to be known but the utility function is unknown to the researcher. The problem is to evaluate whether the change in utility between two arbitrary equilibrium situations (p^0,q^0) and (p^1,q^1) is positive, zero or negative. Revealed preference theory tells that $h(p,C)$ in principle gives sufficient information to solve the problem.

Our operational solution is based on the theory of Divisia-Törnqvist chain indices and consumer surplus measures. We present an algorithm to calculate the compensated income $\bar{C}^1 = C(p^1,q^0)$ and the compensated (or Hicksian) demand $\bar{q}^1 = H(p^1,q^0) = h(p^1,C(p^1,q^0))$ for any (p^1,q^0) using only the known market demand system $q = h(p,C)$. The algorithm is easily interpreted using index number theory or consumer surplus measures and it is shown to work efficiently.

EFFICIENT METHODS OF MEASURING WELFARE CHANGE AND COMPENSATED INCOME IN TERMS OF MARKET DEMAND FUNCTIONS*

1. Introduction

The problem is here formulated in terms of consumer theory. A consumer chooses a bundle of goods $q = (q_1,\ldots,q_n)$ as if he were maximizing a well behaving ordinal utility function $u(q)$ under a budget constraint $p \cdot q = \Sigma p_i q_i \leq C$, where $p = (p_1,\ldots,p_n)$ and $C > 0$ denote exogenous positive prices and expenditure. Let q^0 and q^1 be two equilibrium points corresponding to price - expenditure situations (p^0,C^0) and (p^1,C^1). Our problem is to find out whether the welfare change from q^0 to q^1 is positive ($q^0 \prec q^1$), negative ($q^0 \succ q^1$) or zero ($q^0 \sim q^1$), when the utility function is unknown to us. If we do not know the demand functions $q_i = h^i(p,C), i=1,\ldots,n$, the problem cannot be generally solved. All we can infer if we know only two equilibrium points (p^0,q^0) and (p^1,q^1) is presented in the following revealed preference table, see Vartia (1976), Afriat (1972, p. 20, 1977, p. 64-78).

	$P_q < 1$	$P_q = 1$	$P_q > 1$
$L_q < 1$	$q^1 \prec q^0$	Inconsistent preferences	Inconsistent preferences
$L_q = 1$	$q^1 \precsim q^0$	$q^1 \sim q^0$	Inconsistent preferences
$L_q > 1$	Zone of Indeterminacy	$q^1 \succsim q^0$	$q^1 \succ q^0$

*) I am indebted to L. Törnqvist and P. Vartia for helpful conversations and advice and to H. Vajanne for programming the algorithm.

Here $P_q = Q^P(q^1,q^0,p^1,p^0) = p^1 \cdot q^1/p^1 \cdot q^0$ and $L_q = Q^L(q^1,q^0,p^1,p^0) = p^0 \cdot q^1/p^0 \cdot q^0$ are Paasche's and Laspeyres' quantity indices. Instead of "Inconsistent preferences" we perhaps should write "Impossible under utility hypothesis". Note that if $P_q = Q^P(q^1,q^0,p^1,p^0) > 1$ then $q^1 \succ q^0$ and if $L_q = Q^L(q^1,q^0,p^1,p^0) < 1$ then $q^1 \prec q^0$.

In the Zone of Indeterminacy any two equilibrium points (p^0,q^0) and (p^1,q^1) giving $P_q < 1 < L_q$ could be generated by numerous alternative preferences some of which order q^0 and q^1 differently. This was demonstrated already by Samuelson (1947).

Therefore we have to assume something more to proceed. We suppose that the demand system $h(p,C) = h^1(p,C),\dots,h^n(p,C))$ satisfying the standard utility hypothesis is known to us although the utility function is not. Although the known $h(p,C)$ describes completety the market behaviour of our consumer it is still difficult to evaluate when his satisfaction or utility has increased and when remained constant. The revealed preference theory shows that demand functions give all the information needed to determine the indifference surfaces, see Samuelson (1948 and 1953), Houthakker (1950), Uzawa (1960) and Stigum (1973). The upper and lower sequencies of 'offer curves' used in revealed preference arguments approximate the indifference surface from above and below respectively and converge therefore slowly towards it. Our algorithm generates sequencies of quantity vectors that approximate the indifference surface more accurately and converge quickly towards it. The principle of our algorithm was stated e.g. by Bergson (1975, p. 39).

The approximation of economic index numbers or measurement of consumer surpluses is complicated because of this kind of mainly computational difficulties clearly demonstrated by McKenzie and Pearce (1976). Their theoretically elegant solution to the same problem is based on high order derivates of the demand functions and its applicability depends on how easily these can be evaluated, see also G. McKenzie (1976).

Therefore their solution is as such unsuitable for computer simulation. Our proposed solution presented in our algorithm is based on the theory of Divisia-Törnqvist indices and consumer surplus measures and it is easily taught even to the "labourous full idiot", the computer.

2. Conceptual background

Let $\Omega = \mathbb{R}^{n+1}_+$ be the non-negative quadrant of (n+1) dimensional Euclidean vector space and Ω^* its subset. Consider functions $h: \Omega^* \to \mathbb{R}^n_+$ assigning to any price-expenditure pair (p,C) in Ω^* one and only one quantity vector $q = h(p,C)$ in $\mathbb{R}^n_+$. We are liberal and call $h: \Omega^* \to \mathbb{R}^n_+$ a demand function (or system) if $h(p,C)$ is an element of the budget set $B(p,C) = \{q | p \cdot q \leq C\}$, i.e. if h satisfies BC:

BC. Budget condition: $\forall(p,C)\in\Omega^*$: $p\cdot h(p,C) \leq C$.

We do not consider here demand correspondences or more general choice functions where $h(p,C)$ may denote a set of q's, see Richter (1966). The name 'demand function' is often used only for h's that satisfy

B. Balance: $\forall(p,C)\in\Omega^*$: $p\cdot h(p,C) = C$

H. Homogeneity of degree zero:

$$\forall(p,C)\in\Omega^*:\forall\lambda > o: h(\lambda p,\lambda C) = h(p,C) = h(p/C,1),$$

see e.g. Kihlström, Mas-Colell and Sonnenschein (1976), Shafer (1974). As we said we are more liberal here.

A demand function $h(p,C)$ may or may not correspond to some (utility) function $u: \mathbb{R}^n_+ \to \mathbb{R}$. We say that a utility function $u: \mathbb{R}^n_+ \to \mathbb{R}$ represents a given demand function $h: \Omega^* \to \mathbb{R}^n_+$ if $h(p,C)$ is the unique u-maximal element in any budget set $B(p,C)$: For all $(p,C)\in\Omega^*$: $\forall q\in B(p,C)$: $q \neq h(p,C) \Rightarrow u(q) < u(h(p,C))$. Sometimes such a $u(q)$ is said to 'rationalize' $h(p,C)$. We try to apply here Richter's (1966) terminology, where 'rationalize' is used only in connection with (preference) relations R and is therefore more general.

A demand function may satisfy the following (rather weak) utility hypothesis.

WUH: Weak utility hypothesis

The demand function $h: \Omega^* \to \mathbb{R}^n_+$ is representable by some utility function, i.e. there exists a function $u: \mathbb{R}^n_+ \to \mathbb{R}$ representing the given demand function.

Note that if $u(q)$ represents $h(p,C)$ and $g: \mathbb{R}\to\mathbb{R}$ is strictly increasing then also $\bar{u}(q) = g(u(q))$ represents $h(p,C)$. If $h(p,C)$ satisfies WUH then H is true, but e.g. B may well be untrue. It is a standard practice in demand theory to derive demand systems using a Lagrangian $F(q,\lambda) = u(q) - \lambda(p\cdot q-C)$, where $u(q)$ is sufficiently well-behaving utility function, see e.g. Wold and Jureen (1953), Rajaoja (1958), Malinvaud (1972, p. 12-42), Phlips (1974). Following this line of thinking gives rise to the following (rather strong) utility hypothesis:

SUH: Standard utility hypothesis

The demand function $h: \Omega^* \to \mathbb{R}^n_+$ is representable by a standard (i.e. continuosly twice differentiable, strictly increasing and strictly quasi-concave) utility function $u: \mathbb{R}^n_+ \to \mathbb{R}$.

Our standard utility function is more specific than e.g. the normal utility function of Afriat (1972, p. 32). Between the weak and standard utility hypotheses there are many intermediate cases, which complicates the issue. If SUH holds for a $h(p,C)$ then it satisfies B and H, is continuous and differentiable and has many other nice properties. E.g. its Slutsky matrix $A(p,C)$, a $(n \times n)$-matrix consisting of substitution terms $a^i_j(p,C) = h^i_j(p,C) + h^j(p,C)h^i_{n+1}(p,C)$, is symmetric (S) and negatively semidefinite (NSD) for all $(p,C)\in\Omega^*$. Conditions S and NSD are just the economic integrability conditions considered by Hurwicz (1971), see also Kihlström, Mas-Colell and Sonnenschein (1976) or Chipman and Moore (1976, p. 79 and 111).

As demonstrated first by Hicks (1946) and Samuelson (1947) and shown later by Shephard (1953, 1970) in the contex of production theory, L. McKenzie (1957), Diewert (1971), Afriat (1972, 1977) and others it is possible to define the minimum expenditure (or cost) function

$$C(p,q) = \min\{C \mid C = p\cdot\tilde{q} \ \&\ u(\tilde{q}) = u(q)\} = \min_{\tilde{q}\sim q} p\cdot\tilde{q}$$

under fairly general conditions on $u(q)$. $C(p,q)$ is the minimum expenditure needed to bye the well-being determined by q (i.e. some $\tilde{q}$ indifferent to q) when prices are p. For any given p the function $C(p,q)$ of q is a utility function, in particular $q \sim \bar{q} \Leftrightarrow C(p,q) = C(p,\bar{q})$, see e.g. Afriat (1972, p. 17 and 36). If $q = h(p,C)$ or (p,q) is an equilibrium pair then $C = p\cdot q = C(p,q)$. We will use $C(p,q)$ freely in our later operations and regard as evident that it is sufficiently well-behaving when $u(q)$ is a standard utility function. We suppose in the sequel that our demand system $h(p,C)$ satisfies standard utility hypothesis SUH and has therefore all these nice properties.

This assumption (together with minor technical assumptions) is sufficient for our algorithm to work appropriately. The questions which assumptions would be necessary or what would our algorithm do in more general situations (e.g. in the case of nontransitive consumer, see Shafer (1974)) are left here aside.

3. Compensated income and compensated demand

The problem is further specified as follows. Choose any price-expenditure pair (p^0,C^0) and let $q^0 = h(p^0,C^0)$ be the corresponding unique market demand, where $h(p,C)$ is supposed to be known. Change prices $p^0 \to p^1$ and determine the compensated income (or rather compensating expenditure[1])

$$\text{(1)} \qquad \bar{C}^1 = C(p^1,q^0) = \min_{q\sim q^0} p^1\cdot q$$

$$= \min\{C \mid C = p^1\cdot q \ \&\ u(q) = u(q^0)\}$$

and the Hicksian (or compensated) demand

$$\text{(2)} \qquad \bar{q}^1 = H(p^1,q^0) = h(p^1,C(p^1,q^0))$$

for any given price vector p^1. Of course one of $\bar{q}^1$ and $\bar{C}^1$ determines the other because $\bar{q}^1 = h(p^1,\bar{C}^1)$ and $\bar{C}^1 = p^1\cdot\bar{q}^1$. Here $\bar{q}^1$ is the cheapest bundle of goods under prices p^1,

1) To fix ideas think that prices increase. Then more income is needed to attain the previous level of living or to compensate for the price change: $\bar{C}^1 > C^0$. Here $\bar{C}^1-C^0$ is the compensation (or needed extra income) in monetary units, $100(\bar{C}^1-C^0)/C^0$ in percents and $100\ln(\bar{C}^1/C^0)$ in log-persents, and $\bar{C}^1$ is the compensated income, which includes the compensation. Compensated demand curves are used to define the substitution and income effects. Terminology is rather

which gives the same satisfaction as q^0 and $\bar{C}^1$ is the least expenditure needed to attain the satisfaction given by q^0, when prices have changed to p^1. As our $h(p,C)$ is supposed to satisfy SUH a well-behaving utility function exists but is not known. The compensated income (1) should be determined using only the market demand system $h(p,C)$.

4. Economic price and quantity indices

These quantities allow us to compute e.g. the (Laspeyres' type) economic price index

$$(3) \qquad P(p^1,p^0;q^0) = C(p^1,q^0)/C(p^0,q^0) = \bar{C}^1/C^0$$

and the (Paasche's type) economic quantity index

$$(4) \qquad Q(q^1,q^0;p^1) = C(p^1,q^1)/C(p^1,q^0) = C^1/\bar{C}^1 = (C^1/C^0)/P(p^1,p^0;\ q^0)$$

corresponding to any two equilibrium situations (p^0,q^0) and (p^1,q^1), where $q^t = h(p^t,C^t)$, t=0,1. Because $Q(q^1,q^0;p^1)$ is for fixed q^0 and p^1 and for arbitrary variable q^1 a utility function[1], it solves e.g. our original problem:

1) That is, $Q(q^1,q^0;p^1) = Q(q^2,q^0;p^1)$ if and only if $q^1 \sim q^2$, and $Q(q^1,q^0;p^1) > Q(q^2,q^0;p^1)$ if and only if $q^1 \succ q^2$. This kind of general properties of (3) and (4) following from those of $C(p,q)$ are supposed to be known, see e.g. Samuelson and Swamy (1974), Theil (1975, p. 112-144), Vartia (1976).

$$(5) \qquad \begin{aligned} Q(q^1,q^0;p^1) > 1 &\Leftrightarrow C^1 > \bar{C}^1 \Leftrightarrow q^1 \succ q^0 \\ Q(q^1,q^0;p^1) = 1 &\Leftrightarrow C^1 = \bar{C}^1 \Leftrightarrow q^1 \sim q^0 \\ Q(q^1,q^0;p^1) < 1 &\Leftrightarrow C^1 < \bar{C}^1 \Leftrightarrow q^1 \prec q^0 \end{aligned}$$

Or verbally: if the actual income $C^1 = p^1 \cdot q^1$ exceeds (falls short) the income $\bar{C}^1$ just compensating for the price change $p^0 \to p^1$ then the welfare change from q^0 to q^1 is positive (negative). If $\bar{C}^1 = C^1$ the welfare has remained the same. Next we give some differential expressions stating necessary conditions for movements on the same indifference surface.

5. Conditions for movements on the same indifference surface

Let t denote an auxiliary variable such that $0 \leq t \leq 1$ and let $p(t)$ be a differentiable curve in the price space connecting $p^0 = p(0)$ to $p^1 = p(1)$. $C(t)$ is any expenditure development starting from $C^0 = C(0)$. If $u(q)$ is a possible utility function, then $V(p,C) = u(h(p,C))$ is the corresponding indirect utility function. Derivating $V(t) = V(p(t),\ C(t))$ in respect of t we get

$$(6) \qquad \frac{dV(t)}{dt} = \sum_{i=1}^{n} \frac{\partial V(p(t),C(t))}{\partial p_i(t)} \frac{dp_i(t)}{dt} + \frac{\partial V(p(t),C(t))}{\partial C(t)} \frac{dC(t)}{dt}\ .$$

Using Roy's theorem[1)] we get

$$(7)\qquad \frac{dV(p(t),C(t))}{dt} = \lambda(p(t),C(t))\left[\frac{dC(t)}{dt} - \sum h^i(p(t),C(t))\frac{dp_i(t)}{dt}\right].$$

By the usual assumption $\lambda(p,C) > 0$ of insatiation a necessary and sufficient condition for $h(p(t),C(t))$ moving on the same indifference surface is that (7) equals zero which leads to the first order differential equation in $C(t)$:

$$(8)\qquad \frac{dC(t)}{dt} = \sum h^i(p(t),C(t))\frac{dp_i(t)}{dt}.$$

Note that $p(t)$ and the derivates $dp_i(t)/dt$ are here known functions. Integrating this we get an equivalent integral equation

$$(9)\qquad C(t)-C^0 = \sum\int_0^t h^i(p(t),C(t))\frac{dp_i(t)}{dt}\,dt.$$

Let $p(t)$ be any differentiable price curve connecting p^0 to p^1. By the definition (1) of compensated income $C(t) = C(p(t),q^0)$ the compensated demand $H(p(t),q^0) = h(p(t), C(p(t),q^0))$ moves on the indifference surface determined by q^0 when $t\in[0,1]$ changes. Therefore the compensated income $C(t) = C(p(t),q^0)$ is a solution of both (8) and (9) having the initial value $C(0) = C^0 = p^0 \cdot q^0 = p^0 \cdot h(p^0,C^0)$. Using the uniqueness property of first order differential equations,

1) That is: $\partial V(p,C)/\partial p_i = -\frac{\partial V(p,C)}{\partial C}h^i(p,C) = -\lambda(p,C)h^i(p,C)$. For a short, elegant and very general proof see Chipman and Moore (1976, p. 74).

see e.g. Henrici (1964, p. 264), the compensated income $C(t) = C(p(t),q^0)$ is the only solution having this initial value. Therefore by solving (8) or (9) we get just the compensated income $C(p(t),q^0)$.

Equations (7) and (8) correspond to the usual but somewhat ambiguous total differential expressions $dV = \lambda(dC-\Sigma q_i dp_i)$ and $dC = \Sigma q_i dp_i$, see e.g. Silberberg (1972), Burns (1973), McKenzie and Pearce (1976).

By a simple transformation (8) may be expressed equivalently as

$$(10)\qquad \frac{d\log C(t)}{dt} = \sum w_i(p(t),C(t))\frac{d\log p_i(t)}{dt}, \text{ where}$$

$$w_i(p,C) = p_i h^i(p,C)/C$$

is the i^{th} value share. The integrated version is

$$(11)\qquad \log\frac{C(t)}{C^0} = \sum\int_0^t w_i(p(t),C(t))\,d\log p_i(t).$$

The only solution $C(t)$ starting from $C(0) = C^0 = p^0 \cdot q^0$ is also here the compensated income $C(p(t),q^0)$ corresponding to the given price curve $p(t)$.

Note that when (11) is solved its left hand side is the logarithm of the economic price index (3), $\log[C(t)/C^0] = \log[C(p(t),q^0)/C^0] = \log P(p(t),p^0;q^0)$, and its right hand side is the Divisia-Törnqvist integral representation of $\log P(p(t),p^0;q^0)$. The value shares in (11) are determined from demand $h(p(t),C(t))$ constrained on the same indifference surface.

Note that $\bar{C}^1 = C(p^1,q^0)$ gives the only solution $C(t)$ of equations (8)-(11) for t=1 and for arbitrary price curve $p(t)$ connecting p^0 to p^1. This means that the same compensated income $\bar{C}^1$ results irrespective the choice of an appropriate $p(t)$ curve. If the left hand sides of (9) and (11) are written as line integrals in the (n+1)-dimensional (p,C)-space, these line integrals are independent of the path of integration, when $h(p(t),C(t))$ moves on the same indifference surface. This is shown and discussed e.g. by Silberberg (1972, p. 947-948), Burns (1973, 1977) and Bruce (1977).

6. How to move on the same indifference surface

Our algorithm of calculating $\bar{C}^1 = C(p^1,q^0)$ is based on equations (8)-(9); almost as simple algorithms may be derived from (10)-(11).

Choosing $t_0,t_1,\ldots,t_N$ so that $0=t_0<t_1<\ldots<t_N=1$ we derive from (9) the following

$$(12)\qquad \bar{C}^1 - C^0 = \sum_{k=1}^{N}[C(t_k)-C(t_{k-1})] = \sum_{k=1}^{N}[\Sigma\int_{t_{k-1}}^{t_k} h^i(p(t),C(t))dp_i(t)]\ .$$

The bracketed terms are pairwise equal. Approximating the integrands $h^i(p(t),C(t))$ by the average of their end point values, cf. Collatz (1960, p. 53), we get for $k = 1,2,\ldots,N$

$$(13)\qquad C(t_k) - C(t_{k-1}) \approx \Sigma\tfrac{1}{2}[h^i(p(t_k),C(t_k))+h^i(p(t_{k-1}),C(t_{k-1})](p_i(t_k)-p_i(t_{k-1}))\ .$$

Equations (12)-(13) form the basis of our algorithm. Similar algorithms are derived using other approximations for the integrands or starting from equations (10)-(11).

The compensated income (1) and the Hicksian demand (2) may be calculated simultaneously using the following algorithm.

<u>Algorithm 1:</u> Let $p(t) = p^0+t(p^1-p^0)$, $0 \le t \le 1$, be the linear price curve connecting p^0 to p^1. For a given integer N let $t_k = k/N$, $p_k = p(t_k)$ and generate a sequence $C_1,\ldots,C_N$ so that

$$(14)\qquad C_k - C_{k-1} = \tfrac{1}{2}(q_k + q_{k-1}) \cdot (p_k - p_{k-1}),$$

where $q_k = h(p_k, C_k)$, $k = 1,\ldots,N$ and the starting values are $(p_0,q_0,C_0) = (p^0,q^0= h(p^0,C^0),C^0)$.

The solution C_k of (14) is determined iteratively as follows

$$(15)\qquad C_k^{(m)} = C_{k-1} + \tfrac{1}{2}(q_k^{(m-1)}+ q_{k-1}) \cdot (p_k - p_{k-1}),$$

where $q_k^{(m-1)} = h(p_k, C_k^{(m-1)})$ and $C_k^{(0)} = C_{k-1}$, $k \geq 1$. When $|C_k^{(m)} - C_k^{(m-1)}|$ is negligible set $C_k = C_k^{(m)}$ and $q_k = q_k^{(m)}$ and start the calculation for the next k.

Theorem 1: Under weak conditions given in Appendix C_N converges to the compensated income $\bar{C}^1 = C(p^1, q^0)$ and $q_N = h(p_N, C_N)$ converges to the compensated demand $\bar{q}^1 = H(p^1, q^0) = h(p^1, \bar{C}^1)$ as N increases. The convergence is cubical, i.e. errors decrease in relation to $(1/N)^3$.

The theorem and convercenge[1] of (15) are proved in appendix. The idea of the algorithm is to move by small steps in the indifference surface from q^0 to $\bar{q}^1$. Each q_k approximates $\bar{q}_k = H(p_k, q^0)$, the true compensated demand corresponding to p_k and q^0. Equation (14) is an accurate discrete analog for equation (8). Actually (14) requires that (p_k, q_k) and (p_{k-1}, q_{k-1}) are two equilibrium points, for which the Harberger welfare indicator (see Harberger (1971), Diewert (1976))

1) Note that a practical way of writing (15) is

$$(15') \qquad C_k^{(m)} = \tfrac{1}{2} q_k^{(m)} \cdot (p_k - p_{k-1}) + C^*,$$

where $C^* = C_{k-1} + \tfrac{1}{2} q_{k-1} \cdot (p_k - p_{k-1})$ is independent of m.

$$(16) \qquad H(p_{k-1}, p_k, q_{k-1}, q_k) = p_{k-1} \cdot (q_k - q_{k-1}) + \tfrac{1}{2}(p_k - p_{k-1}) \cdot (q_k - q_{k-1})$$

$$= \tfrac{1}{2}(p_k + p_{k-1}) \cdot (q_k - q_{k-1})$$

is zero. To show this we only need to note that $H(p^1, p^2, q^1, q^2) = p^1 \cdot (q^2 - q^1) + \tfrac{1}{2}(p^2 - p^1) \cdot (q^2 - q^1) = \tfrac{1}{2}(p^2 + p^1) \cdot (q^2 - q^1)$ is zero if and only if

$$(17) \qquad C^2 - C^1 = \tfrac{1}{2}(q^2 + q^1) \cdot (p^2 - p^1) = H(q^1, q^2, p^1, p^2)$$

where $C^2 = p^2 \cdot q^2$ and $C^1 = p^1 \cdot q^1$. Equation (17) says approximately that the change in expenditure is all needed to compensate for the price changes. Generally, change in expenditure has a decomposition

$$(18) \qquad C^2 - C^1 = \tfrac{1}{2}(p^2 + p^1) \cdot (q^2 - q^1) + \tfrac{1}{2}(q^2 + q^1) \cdot (p^2 - p^1)$$

$$= H(p^1, p^2, q^1, q^2) + H(q^1, q^2, p^1, p^2)$$

into arithmetic contributions of quantity and price changes. Note that this is the finite change version of $dC = \Sigma p_i dq_i + \Sigma q_i dp_i$. Therefore $H(p^1, p^2, q^1, q^2) = 0$ if and only if (17).

The decomposition (18) was the starting point of Stuvel (1957) to derive his remarkable price and quantity indices. Stuvel's quantity index has e.g. the representation

(19) $$Q^S(q^2;q^1,p^2,p^1) = A + \sqrt{A^2 + C^2/C^1}, \quad \text{where}$$

$$A = \frac{1}{2}\left(\frac{p^1\cdot q^2}{p^1\cdot q^1} - \frac{p^2\cdot q^1}{p^1\cdot q^1}\right) = \frac{1}{2}(L_q - L_p) = \frac{1}{2}\left(L_q - \frac{C^2/C^1}{P_q}\right).$$

Stuvel's index satisfies the time and factor reversal tests, reacts correctly to extreme quantity or price changes, is consistent in aggregation and has other remarkable properties, see Stuvel (1957), van Yzeren (1958), Banerjee (1975) and Vartia (1976, p. 140, 159-172). Van Yzeren shows e.g. that (19) and Edgeworth's quantity index $Q^E(q^2,q^1,p^2,p^1) = (p^2+p^1)\cdot q^2/(p^2+p^1)\cdot q^1$ equal one together. We see at once that this happens exactly if $H(p^1,p^2,q^1,q^2) = 0$ or equivalently if (17) holds. These expressions are beatifully symmetric and easy to work with. These facts enable us to say that (14) requires (when trying to remain on the same indifference surface where the economic quantity index is identically one) that we choose our small steps so that the following two conditions are satisfied for all $k = 1,\dots,N$:

(C1) The quantity vector q_k is the demand corresponding to prices p_k and expenditure C_k:

$$q_k = h(p_k,C_k) = h(p_k,p_k\cdot q_k).$$

(C2) Stuvel's (or equivalently Edgeworth's) quantity index comparing consequtive pairs $(p_{k-1},q_{k-1}),(p_k,q_k)$ remains equal to one:

$$Q^S(q_k,q_{k-1},p_k,p_{k-1}) = 1.$$

Similar conditions using other index numbers and approximations of demand functions (or Engel curves) appear in approximating the economic or true price index, see. e.g. Frisch (1936), Wald (1939) or Banerjee (1975) although notation sometimes hides the principles[1]. Banerjee (1975, p. 96-109) uses ecplicitely Stuvel's index in his "factorial approach" but demand functions do not appear explicitely. If pairs (p_k,q_k) are observations from "demand world" then C1 holds automatically, which is not necessarily true if the researcher generates them.

Thus our chain of quantity index calculations multiplies into one. We have, approximately, followed a path of equilibrium points, where the logarithm of the Divisia-Törnqvist quantity index, see Samuelson and Swamy (1974) and Vartia (1976),

(20) $$\sum\int_0^t w_i(t)\,d\log q_i(t) = \sum_{i=1}^{n}\int_0^t w_i(p(t),C(t))\,d\log h^i(p(t),C(t))$$

---

1) The considerations are intimately connected with e.g. the concepts of consumer surplus, compensated and equivalent income variations and different Divisia-Törnqvist line integrals, which provide alternative more or less different means to handle problems. But these are often used too freely (arguments are omitted etc). Notable recent articles against or in favour of some use of these measures are e.g. Bergson (1975), Bruce (1977), Burns (1973, 1977), Chipman and Moore (1976), Diewert (1976), Foster and Neuberger (1974), Harberger (1971), G. McKenzie (1976), McKenzie and Pearce (1976) and Silberberg (1972). We think that the things would become clearer if the different measures were discussed in relation to economic price and quantity indices $P(p^1,p^0:q^*)$ and $Q(q^1,q^0,p^*)$, where q^* and p^* are some reference quantities and prices, see Samuelson and Swamy (1974) and Vartia (1976). It is a sad fact that only in simple homothetic cases these functions are independent of q^* and p^*. This is one but only one source of confusion.

which for all price-expenditure developments $(p(t);C(t))$ starting from (p^0,C^0) is identical with

$$(21)\qquad \log(C(t)/C^0) - \sum\int_0^t w_i(t)\,d\log p_i(t) =$$

$$\log(C(t)/C^0) - \sum_{i=1}^{n}\int_0^t w_i(p(t),C(t))\,d\log p_i(t)\,,$$

has remained equal to zero. This is a sufficient[1] condition for movements on an indifference surface. Note that here is no trouble of the possible path dependency of the Divisia-Törnqvist line integral because for paths on the same indifference surface it is path-independent, i.e. only end points matter. Therefore any convenient price-path in from p^0 to p^1 may be used.

These economic considerations led to the invention of our algorithm. Mathematically the algorithm happens to be a special case of Adams interpolation method for numerical solution of differential equations, which is used in proving that it works efficiently, see the appendix.

1) It is also necessary if $\lambda(p,C)$, the marginal utility of expenditure, is positive.

If the same method is used to solve the differential equation (10) in logarithms we get the Algorithm 2, where (14) is replaced by

$$(22)\quad \log(C_k/C_{k-1}) = \sum_{i=1}^{n}\tfrac{1}{2}(w_i(p_k,C_k)+w_i(p_{k-1},C_{k-1}))\log(p_{k,i}/p_{k-1,i})$$

$$= \log P^T(p_k,p_{k-1},q_k,q_{k-1}).$$

Here we have the Törnqvist's price index for which

$$(23)\qquad \log P^T(p^1,p^0,q^1,q^0) = \sum\tfrac{1}{2}(w_i^1 + w_i^0)\log(p_i^1/p_i^0)$$

$$= \sum_{i=1}^{n}\tfrac{1}{2}\left(\frac{p_i^1q_i^1}{p^1\cdot q^1} + \frac{p_i^0q_i^0}{p^0\cdot q^0}\right)\log(p_i^1/p_i^0).$$

Algorithm 2 works perhaps still better than Algorithm 1, because value shares $w_i = w_i(p,C) = p_i h^i(p,C)/C$ are usually more slowly changing characteristics than quantities $q_i = h^i(p,C)$.

As in (14) iteration is also needed in (22) to solve C_k. Theorem 1 renamed as Theorem 2 is proved similarly for Algorithm 2.

Using other price index number formulas instead of (23) we get other algorithms. It is intuitively clear that convergence properties are not altered if Törnqvist's index is replaced by Vartia-Sato index

$$\text{(24)} \qquad \log P^{VS}(p^1,p^0,q^1,q^0) = \sum_{i=1}^{n} \frac{L(w_i^1,w_i^0)}{\Sigma L(w_j^1,w_j^0)} \log(p_i^1/p_i^0)$$

where $L(x,y) = (x-y)/\log(x/y)$ is the logarithmic mean of positive x and y, see Vartia (1974, 1976, 1976b) and Sato (1974, 1976). Evidently any quadratic approximation of (23) and (24) for small relative changes in p's and q's, such as Fisher's ideal index, Diewert-Sato quadratic mean of order r indices (see Diewert (1974, 1975, 1975b), Sato (1974), Vartia (1978)), Stuvel's or Edgeworth's indices used in Algorithm 1, or just any good approximations of these indices, could be used to define a good 'substitute' for Algorithm 2. Note that Laspeyres' or Paasche's indices are not sufficiently good approximations of these indices and using them in place of (23) will slow down the convergence, cf. Algorithm 3 in Appendix 1.

Using other efficient numerical methods (which are numerous, see e.g. Collatz (1960, p. 536)) to solve differential equations (8) or (10) leads to other efficient algorithms to calculate compensated income and compensated demand.

It is an easy task for a competent ADP specialist to program the Algorithm 1 for any computer[1]. Calculation can even be carried out using only paper, pencil and a functional pocket calculator as shown below.

1) A program written in GE 635 FORTRAN IV is available upon request.

7. Illustrative calculations

It is convenient to present the calculations in a table, where columns are reserved for vectors p_k and $q_k^{(m)}$ and for scalar $C_k^{(m)}$. We illustrate the algorithm using the simple example of McKenzie and Pearce (1976), where $h(p,C) = (\frac{p_2}{p_1}(\frac{C}{p_1+p_2}), \frac{p_1}{p_2}(\frac{C}{p_1+p_2}))$. The demand system corresponds to the "unknown" indirect utility function $V(p,C)=C/p_1+C/p_2$, which we are not allowed to use here. The two equilibrium points are given in Table 1.

Table 1.

Variable	p		q		C
(0) Initial values	1.0000	2.0000[1]	146.6667	36.6667	220.0000
(1) Final values	1.1000	1.6923	121.2119	51.2125	220.0000

You who know the utility function can check that the change in satisfaction is zero, or q^0 and q^1 lay on the same indifference surface.

We start from the initial situation (p^0,q^0,C^0), try to move step by step on the indifference surface and approach the point of compensated demand $\bar{q}^1 = H(p^1,q^0)$, which here is equal to $q^1 = h(p^1,C^1) = (121.2119, 51.2125)$. Let us first use 4 steps, i.e. N = 4.

1) McKenzie and Pearce (1976) have a missprint here.

The calculations run as follows: First calculate the linear price path $p^0 = p_0, p_1, p_2, p_3, p_4 = p^1$ given in Table 2. In the first row (k = 0) we have the starting values $(p^0, q^0, C^0) = (p_0, q_0, C_0)$. Using the demand system $q = h(p,C)$ calculate then for $(k,m) = (1,1)$ $q_1^{(1)} = h(p_1, C_1^{(0)}) = h(p_1, C_0) = (140.0092, 39.7751)$. Next form the average $\frac{1}{2}(q_1^{(1)} + q_0)$, store it somewhere and take the inner product $\frac{1}{2}(q_1^{(1)} + q_0) \cdot (p_1 - p_0)$, which gives $C_1^{(1)} = 220.6433$. This is a new start and the next row is generated similarly: $q_1^{(2)} = h(p_1, C_1^{(1)}), C_1^{(2)} = C_0 + \frac{1}{2}(q_1^{(2)} + q_0) \cdot (p_1 - p_0)$. The iteration for $C_1^{(m)}$ converges quickly and after its convergence calculations for k = 2 proceed completely in the same way.

Table 2. Demand system: $q = h(p,C) = \left(\frac{p_2}{p_1}\left(\frac{C}{p_1+p_2}\right), \frac{p_1}{p_2}\left(\frac{C}{p_1+p_2}\right)\right)$

Price steps: $p_k - p_{k-1} = (0.025, -0.076925)$

				Approximations for the		
		Price situation		compensated demand		compensated income
k	m	p_k		$q_k^{(m)}$		$C_k^{(m)}$
0		1.0000	2.0000	146.6666	36.6666	220.0000
1	1	1.0250	1.9231	140.0092	39.7751	220.6433
	2			140.4186	39.8915	220.6439
	3			140.4190	39.8916	220.6440
2	1	1.0500	1.8462	133.9518	43.3305	220.8727
	2			134.0907	43.3754	220.8727
3	1	1.0750	1.7692	127.8064	47.1849	220.6632
	2			127.6852	47.1402	220.6634
	3			127.6853	47.1402	220.6634
4	1	1.1000	1.6923	121.5774	51.3669	219.9904
	2			121.2066	51.2103	219.9918
	3			121.2074	51.2106	219.9917

The five points $q^0 = q_0, q_1, q_2, q_3, q_4$ lie very near the same indifference surface and $q_4 = (121.2074, 51.2106)$ accurately approximates $\bar{q}^1 = H(p^1, q^0) = (121.2119, 51.2125)$. The economic price index $P(p^1, p^0; q^0) = \bar{C}^1/C^0$ (which equals 1 here) is estimated by $C_4/C^0 = 219.9917/220 = 0.99996$ and the economic quantity index $Q(q^1, q^0; p^1) = C^1/\bar{C}^1$ (which also equals 1 here) is estimated by $C^1/C_4 = 220/219.9917 = 1.00004$. Anyone who does not regard these estimates accurate enough may increase the accuracy without limits by increasing the number of steps from 4. It is convenient e.g. to half the price steps, or in some other way go through the previous price situations. This makes it possible to check the calculations and control the convergence.

Omitting the figures referring to the iteration steps and tabulating only the converged values we get for N = 8 steps the following table, where also the economic price index $P(p_k, p^0; q^0) \approx C_k/C^0$ comparing price situation p_k to the initial prices is included.

Table 3. Demand system: $q = h(p,C) = (\frac{p_2}{p_1}(\frac{C}{p_1+p_2}),\ \frac{p_1}{p_2}(\frac{C}{p_1+p_2}))$

Price steps: $p_k - p_{k-1} = (0.0125,\ -0.0384625)$

				Approximations for the			
		Price situation		compensated demand		compensated income	economic price index
k	m	p_k		$q_k^{(m)} = q_k$		$c_k^{(m)} = c_k$	$P(p_k,p^0;q^0)$
0		1.0000	2.0000	146.6667	36.6667	220.0000	1.00000
1	3	1.0125	1.9615	143.5535	38.2482	220.3732	1.00170
2	3	1.0250	1.9231	140.4198	39.8918	220.6453	1.00293
3	2	1.0375	1.8846	137.2260	41.6002	220.8136	1.00370
4	2	1.0500	1.8462	134.0924	43.3759	220.8754	1.00398
5	2	1.0625	1.8077	130.8994	45.2219	220.8278	1.00376
6	3	1.0750	1.7692	127.6878	47.1412	220.6677	1.00304
7	3	1.0875	1.7308	124.4580	49.1367	220.3921	1.00178
8	3	1.1000	1.6923	121.2106	51.2119	219.9976	0.99999

All quantity vectors of Table 3 lie practically on the same indifference surface. Every second row of table 3 correspond to a row of table 2, which makes it possible e.g. to control the convergence.

Using 4 steps we ended to the approximation $H(p^1,q^0) \approx (121.2074,\ 51.2106)$ as 8 steps gave $H(p^1,q^0) \approx (121.2106,\ 51.2119)$. The price steps are rather long even here, as for the second commodity they are about 2 %. However, the accuracy is sufficient for most purposes.

In computer simulations perhaps only the last row of tables such as 2 or 3 corresponding to $H(p^1,q^0)$ deserves to be printed.

As a final illustration let $p^2 = (1.0500,\ 1.8462)$ and $C^2 = 221.0000$. The demand system gives the quantity vector $q^2 = h(p^2,C^2) = (134.1693,\ 43.3985)$, which the consumer would bye in this situation. Is the consumer better off in situation (2) than in situation (0) of table 1?

From table 3 (row k = 4) we see that $P(p^2,p^0;q^0) \approx 1.0040$ or that 0.40 % more money is needed in situation (2) to compensate for the price increase. Expenditure has increased actually from 220 to 221 or 0.46 %. Hence real consumption has increased somewhat (about 0.06 %) and the consumer lies on a higher utility level. Table 2 gives the same results.

8. Conclusions

We have considered a demand system $h(p,C)$ satisfying the standard utility hypothesis SUH, i.e. which is representable by some standard utility function $u(q)$. An efficient algorithm is presented to calculate the compensated income $C(p^1,q^0) = \min\{p^1\cdot q \mid u(q) = u(q^0)\}$ and the compensated or Hicksian demand $H(p^1,q^0) = h(p^1,C(p^1,q^0))$ as accurately as one wishes using only the known market demand system $h(p,C)$. A well-behaving utility function $u(q)$ exists by SUH but is not used nor needed in the calculation. Using the compensated income $C(p^1,q^0)$ we may compute the 'true' or 'economic' price index (of the Laspeyres' type) $P(p^1,p^0;q^0) = C(p^1,q^0)/p^0\cdot q^0$ and its pair, the 'economic' quantity index (of the Paasche's type)

$Q(q^1,q^0;p^0) = p^1 \cdot q^1/C(p^1,q^0)$ for any two equilibrium points (p^0,q^0) and (p^1,q^1). In fact the price index $P(p^1,p^0;q^0)$ may be calculated by our method for any p^1. But to determine $Q(q^1,q^0;p^1)$ for a given quantity vector q^1 we have to find first some price vector p^1 satisfying $q^1 = h(p^1,p^1 \cdot q^1)$. Of course, if p^1 is a solution, also λp^1 is one for any $\lambda > 0$. This calls for the inverse demand function $r = \psi(q)$, where $r = p/C$, see e.g. Chipman and Moore (1976, p. 104). Alternatively we may use some numerical method to solve $q^1 = h(r^1,1)$ for r^1 and put $p^1 = \lambda r^1$ for some $\lambda > 0$.

Starting from (p^1,q^1) instead of (p^0,q^0) and using the time reversal relations

(25) $$P(p^0,p^1;q^1) = 1/P(p^1,p^0;q^1)$$

$$Q(q^0,q^1;p^0) = 1/Q(q^1,q^0;p^0)$$

we may calculate similarly another pair of indices $P(p^1,p^0:q^1) = p^1 \cdot q^1/C(p^0,q^1)$, $Q(q^1,q^0;p^0) = C(p^0,q^1)/p^0 \cdot q^0$, see e.g. Samuelson and Swamy (1974) or Vartia (1976). These give another decomposition for the expenditure ratio

(26) $$\frac{p^1 \cdot q^1}{p^0 \cdot q^0} = \frac{p^1 \cdot h(p^1,p^1 \cdot q^1)}{p^0 \cdot h(p^0,p^0 \cdot q^0)} = P(p^1,p^0;q^1)Q(q^1,q^0;p^0).$$

Index numbers of prices and quantities and different measures of consumer surpluses have great intuitive appeal to economists and they are applied constantly. Many of these applications

are not warranted by economic theory and some are clearly misuses as demonstrated by many notable recent articles. However, as our algorithm is based on the Divisia-Törnqvist theory of chain indices and on a consumer surplus measure it provides an example of how these measures can be used also outside the very restrictive case of homothetic preferences.

Appendix 1: Proof of Theorem 1.

For the proof we need some results from the numerical solution of differential equations, see Collatz (1960, pp. 48-114, 536) or Henrici (1964, pp. 263-288). Let $f(t,C)$ be a real valued function defined for $t \in [a,b]$ and for all real C and consider a first order differential equation

$$(1) \qquad C' = f(t,C).$$

Equation (1) symbolizes the following problem, see Henrici (1964, p. 263): Find a function $C = C(t)$, continuous and differentiable for all $t \in [a,b]$, such that

$$(2) \qquad C'(t) = f(t, C(t))$$

for all $t \in [a,b]$.

Let N be a positive integer and $t_k = a+k(\frac{b-a}{N})$, so that $t_0 = a$ and $t_N = b$, $t_k - t_{k-1} = (b-a)/N$ is often called the step length or step and denoted by h. A simple but rather crude numerical method of solving (1) is the "polygon method", where the exact solution $C(t)$ for points $t = t_0, t_1, \ldots, t_N$ is approximated by values $C_0, C_1, \ldots, C_N$ calculated by the formula

$$(3) \qquad C_k = C_{k-1} + (\frac{b-a}{N})\, f_{k-1},$$

where $f_{k-1} = f(t_{k-1}, C_{k-1})$. Collatz (1960, pp. 53-59) proves that if a Lipschitz condition is satisfied the error $C_k - C(t_k)$ tends to zero linearly, i.e. like $1/N$, as the step $(b-a)/N \to 0$.

We apply the polygon method for the differential equation (8) in the text

$$(4) \qquad \frac{dC(t)}{dt} = \Sigma h^i(p(t), C(t)) \frac{dp_i(t)}{dt},$$

where the price path connecting p^0 and p^1 is linear, $p(t) = p^0 + t(p^1 - p^0)$, $0 \leq t \leq 1$.

We have $dp_i(t)/dt = (p_i^1 - p_i^0)$ so that for $p(t) = p^0 + t(p^1 - p^0)$

$$(5) \qquad f(t,C) = \Sigma h^i(p(t),C)(p_i^1 - p_i^0)$$

$$= h(p(t),C) \cdot (p^1 - p^0).$$

The equation (3) becomes Algorithm 3:

$$(6) \qquad C_k = C_{k-1} + \frac{1}{N} h(p_{k-1}, C_{k-1}) \cdot (p^1 - p^0)$$

$$= C_{k-1} + q_{k-1} \cdot (p_k - p_{k-1}),$$

where $p_k = p(t_k)$ and $q_{k-1} = h(p_{k-1}, C_{k-1})$. Here C_k converges linearly to the solution $C(t_k)$ of (4), when the step $1/N$ and therefore the price steps $p_k - p_{k-1} = (p^1 - p^0)/N$ approach zero. This slowly converging algorithm corresponds to Samuelsons (1948) "Cauchy-Lipschitz" approximation. Here C_k's approach the compensated income curve $C(t)$ from above.

A more efficient method for integrating (1) is Adams interpolation method of order 1 which in the notation of Collatz (1960, p. 85 and 536) is presented by

$$(7) \qquad y_{r+1} = y_r + h(f_{r+1} - \frac{1}{2}\nabla f_{r+1})$$

$$= y_r + h\,(\frac{f_{r+1} + f_r}{2})$$

and in our notation by

$$(8) \qquad C_k = C_{k-1} + (\frac{b-a}{N})\,(\frac{f_k + f_{k-1}}{2}).$$

It may be proved, see Henrici (1964, pp. 280-3), that the error $C_k - C(t_k)$ vanishes cubically, i.e. like $(1/N)^3$, as the step $(b-a)/N \rightarrow 0$. A sufficient condition for the convergence is that $f(t,C)$ satisfies the Lipschitz condition, see Henrici (1964, p. 264): There exist a constant L such that for any y,z and all $t \in [a,b]$

$$(9) \qquad |f(t,y) - f(t,z)| \leq L|y-z|.$$

This is a very weak condition which is satisfied e.g. if the derivate $\frac{d}{dC} f(t,C)$ exists and is bounded by L for all $t \in [a,b]$.

Applying the Adams interpolation method (8) to equation (4) with $p(t) = p^0 + t(p^1 - p^0)$ leads to the following equation

$$(10) \qquad C_k = C_{k-1} + \frac{1}{2}\,(q_k + q_{k-1}) \cdot (p_k - p_{k-1}),$$

where $p_k = p(k/N)$, $q_k = h(p_k, C_k)$ and $k = 1,\ldots,N$. While in equation (6) the price change $p_k - p_{k-1}$ was weighted by the "old basket" $q_{k-1} = h(p_{k-1}, C_{k-1})$, we have here the mean basket $\frac{1}{2}(q_k + q_{k-1})$.

Equation (10) is equivalent to equation (14) of our Algorithm 1. The unknown C_k contained on both sides of the equation is determined by iteration as shown in (15), cf. also Collatz (1960, p. 86). The convergence of the iteration is considered in Appendix 2.

We conclude that Algorithm 1 for solving (4) corresponds exactly to Adams interpolation method of order 1. Therefore Alogorithm 1 converges cubically, i.e. $C_k - C(t_k)$ vanishes like $(1/N)^3$, as 1/N and the price steps $p_k - p_{k-1} = (p^1 - p^0)/N$ approach zero. A sufficient condition for the convergence is that $f(t,C) = h(p(t),C) \cdot (p^1 - p^0)$ satisfies the Lipschitz condition[1)] (9) or (which is somewhat over-restictive)

$$(11) \qquad \frac{d}{dC} f(t,C) = \Sigma \frac{d}{dC} h^i(p(t),C)\,(p_i^1 - p_i^0)$$

is bounded by some L for all $t \in [0,1]$. If e.g. all the "income elasticities" $d\log h^i(p(t),C)/d\log C$ are bounded by e and $M = \max|(p_i^1 - p_i^0)/p_i(t)|$ when $t \in [0,1]$ we have, cf. Appendix 2,

$$(12) \qquad |\frac{d}{dC} f(t,C)| = |\Sigma \frac{d\log h^i(p(t),C)}{d\log C}\,(\frac{h_i}{C})\,(p_i^1 - p_i^0)|$$

$$\leq |\Sigma e| \frac{h^i(p(t),C)\,p_i(t)}{C}| \, |\frac{p_i^1 - p_i^0}{p_i(t)}|$$

$$\leq e \cdot M$$

1) Note that this is just the condition 6. of Stigum (1973, p. 412).

so that $L = eM$ works as a Lipschitz constant. It is difficult to imagine cases where the Lipschitz condition is not satisfied. Therefore the convergence is quaranteed in most practical cases.

Especially for $t = t_N = 1$ C_N approaches $C(t_N) = C(1)$ (=the compensated income $\bar{C}^1 = C(p^1,q^0)$ as discussed in the text) and therefore $q_N = h(p^1,C_N) \rightarrow h(p^1,\bar{C}^1) = \bar{q}^1$, the compensated demand, when $1/N \rightarrow 0$. Theorem 1 is proved.

Appendix 2: Convergence of the iteration over m in Algorithm 1.

Iteration (15) over m is the ordinary cob-web-iteration $x_m = f(x_{m-1})$, $m = 1,2,...,$ where

$$(1) \qquad f(x) = C_{k-1} + \frac{1}{2}(h(p_k,x) + q_{k-1}) \cdot (p_k - p_{k-1}).$$

A sufficient condition for its convergence to a unique solution $x = f(x)$ for all starting values $x_0 \in [a,b]$, is that $f(x)$ is differentiable and

$$(2) \qquad |f'(x)| \leq L \text{ for all } x \in [a,b],$$

where L is some constant smaller than 1, see e.g. Henrici (1964, pp. 61-66). Derivating (1) we get

$$(3) \qquad f'(x) = \frac{1}{2} \Sigma \frac{\partial}{\partial x} h^i(p_k,x)(p_{k,i} - p_{k-1,i})$$

$$= \frac{1}{2} \Sigma \frac{\partial \log h^i}{\partial \log x} (\frac{h_i}{x}) (p_{k,i} - p_{k-1,i}).$$

Let $M = \max_i |(p_{k,i} - p_{k-1,i})/p_{k,i}|$ be the greatest relative price change and choose a constant e so that all the income elasticities

$$(4) \qquad e_i(p_k,x) = \partial \log h^i(p_k,x)/\partial \log x$$

are bounded by e, $|e_i(p_k,x)| \leq e$, when $x \in [a,b]$. Then

$$(5) \qquad |f'(x)| \leq \frac{1}{2} \Sigma |e_i(p_k,x)| \left|\frac{p_{k,i} h^i}{x}\right| \left|\frac{p_{k,i} - p_{k-1,i}}{p_{k,i}}\right|$$

$$\leq \frac{eM}{2} \Sigma \frac{p_{k,i} h^i(p_{k,i},x)}{x} = \frac{eM}{2}$$

because of the budget constraint. Therefore (2) is satisfied when only M is chosen sufficiently small.

E.g. if $e = 5$ then choosing all the relative price changes less than $\pm$ 0.4 or 40 % is sufficient to quarantee the convergence.

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