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ASYMPTOTIC INFERENCE IN REGRESSION MODELS WITH AUTOREGRESSIVE ERRORS HAVING ROOTS ON THE UNIT CIRCLE*

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Abstract

An estimation and inference procedure is proposed for time series regression models with stationary regressors and nonstationary autoregressive errors having roots on the unit circle. It is shown that the usual GLS or Cochrane-Orcutt procedure should be done in reverse order, namely by starting the estimation from the error structure. The reason for this is the fact that the nonstationary error part dominates the stationary regression part so strongly that consistent estimation of the nonstationary factor in the error model can be done with fast convergence simply by disregarding the regression part. The proposed three-step ordinary least squares estimation procedure will yield estimates of the unknown parameters of the regression part and the error part having known asymptotic distributions. In particular, the estimators of the error part can be used to test for the existence of roots on the unit circle.

1. Introduction

In this note we will consider a simple time series regression model with autoregressive error terms of the form

$$\phi(B)(y_t - \beta x_t) = a_t \tag{1.1}$$

or equivalently

$$y_t = \beta x_t + \varepsilon_t, \quad \text{where } \phi(B)\varepsilon_t = a_t \tag{1.2}$$

$\phi(B)$ is the usual autoregressive operator $\phi(B) = 1-\phi_1 B-\phi_2 B^2-\ldots-\phi_p B^p$.

If ε_t is stationary, that is, $\phi(B)$ has all its zeroes outside the unit circle, then the usual generalized least squares (GLS) or Cochrane-Orcutt (CO) procedure can be used to obtain estimates of β and the parameters of $\phi(B)$. These procedures start by first doing OLS to obtain an initial estimate of β, then estimating the parameters of the error structure from the regression residuals and finally doing the GLS or CO transformation of y_t and x_t and re-estimating β. The fundamental prerequisite for this procedure to work is that the true error structure is stationary which makes it possible to estimate β consistently at the initial stage. If ε_t is not stationary the OLS estimate of β will not be consistent, unless x_t has at least a linear trend in it.

Our purpose is to consider situations, where the nonstationarity in the model arises solely from the regression error term ε_t. Therefore we want to interpret the model in such a way that the level of y_t is explained by a stationary x_t, and assume that the deviations from this level are allowed to have nonstationary behavior. We will confine ourselves to situations where the nonstationarity arises from $\phi(B)$ having zeroes on the unit circle. Therefore no explosive cases will be discussed. If the deviations show a random walk type of behavior, $\phi(B)$ has a $1-B$ factor. If the deviations have a smooth cyclical pattern, $\phi(B)$ may contain a factor $1-\phi B+B^2$, where $|\phi| < 2$, which has a pair of complex roots on the unit circle.

We will not consider problems of identifying the order of the error autoregression. The reader is referred to a recent article by Tsay (1984) on this issue. For expository simplicity we will first carry out the discussion by not allowing the autoregressive operator to contain any stationary factors. Discussion of the mixed nonstationary-stationary autoregressive operator is postponed until the end of the paper. The important new issues will, however, all become apparent already within the simpler models.

The basic thrust of this note is that if we wish to make inference about the parameters of $\phi(B)$ and β, for example to test for the existence of roots of $\phi(B)$ being on the unit circle, the estimation of the parameters of the model should be started from the dominating nonstationary error structure by regular autoregressive fitting of the y_t observations. Then estimate β using GLS or CO transformation and finally re-estimate $\phi(B)$ from the regression residuals. Therefore the usual estimation procedure should be done in <u>reverse</u> order. It is shown that the three-step ordinary least squares estimates so obtained have known asymptotic distributions under the null hypothesis of nonstationarity of the errors. Specifically, if the null has 1-B as the nonstationary operator, then the autoregressive coefficient has the asymptotic distribution of Dickey & Fuller (1979) and Evans & Savin (1981). If the null has $1-\phi B+B^2$ as the nonstationary operator, then the autoregressive coefficients have the asymptotic distributions of Ahtola & Tiao (1985a). These distributional results can be used to test for the existence of roots on the unit circle. The estimator of β has the usual asymptotic normal distribution. A computationally convenient feature of the proposed estimation procedure is that only ordinary least squares routines are needed at each step.

2. The model and the results

Consider a simple regression model

$$y_t = \beta x_t + \varepsilon_t, \quad t = 1,2,\ldots,n \tag{1.3}$$

where x_t is assumed to be stationary with absolutely summable autocovariance function and

$$\phi(B)\varepsilon_t = a_t, \tag{1.4}$$

where $a_t \sim nid(0,\sigma^2)$, $t = 1,2,\ldots,n$.

We further assume that all the x_t's are independent of the "innovations" $a_1, a_2, \ldots, a_n$ which drive the error process.

The fundamental distinction between model (1.3) with stationary errors and model (1.3) with nonstationary errors, at least as far as the estimation of the unknown parameters is concerned, lies with the dominance of the error part over the regression part, when the error process is nonstationary. This dominance is strong enough to allow us to estimate, with very fast consistency, the parameters of the error process simply by neglecting the regression part altogether. The following lemma summarises this preliminary result.

Lemma 2.1. Let the model be (1.3), (1.4) with the assumptions associated with them. If the error process is of purely nonstationary type with no multiple roots of $\phi(B) = 0$, then ordinary least squares regression in $\phi(B)y_t = \text{error}_t$ yields estimates, $\tilde{\phi}_i$ of the coefficients of $\phi(B)$ such that $\tilde{\phi}_i = \phi_i + O_p(n^{-1})$.

Proof. See the Appendix A.

This Lemma is similar to Corollary 2.1. in Tsay (1984), except that the convergence rate in Lemma 2.1. is faster than that implied by Tsay. His convergence rate is $O_p(n^{-1/2})$ if translated to our case. We should, however, note that Tsay assumes the regressor variable to have deterministic values, whereby he cannot utilize the stochastic nature of the regressor. Suffice it here to say that convergence rate of $O_p(n^{-1/2})$ in Lemma 2.1. would not enable us to derive the properties of the estimators suggested in this paper.

The next Theorem summarises the properties of the GLS or CO estimator of β based on the accordingly transformed y_t and x_t variables.

Theorem 2.1. Let $\tilde{\phi}_i$ be the estimator defined in Lemma 2.1. and $\hat{y}_t$ and $\hat{x}_t$ be the GLS or CO transformed variables (e.g. if p = 1, then $\hat{y}_t = y_t - \tilde{\phi}_1 y_{t-1}$ and $\hat{x}_t = x_t - \tilde{\phi}_1 x_{t-1}$). Then regression of $\hat{y}_t$ on $\hat{x}_t$ results in the OLS estimator of β, $\hat{\beta}$, which is consistent and has the asymptotic normal distribution of $\sqrt{n}(\hat{\beta}-\beta) \xrightarrow{L} N(0,\sigma^2\gamma^{-2})$, where $\gamma = \text{plim}\ \frac{1}{n} \Sigma\hat{x}_t^2$.

Proof. See the Appendix B.

Now the natural next step is to use the residuals $y_t - \hat{\beta}x_t$'s to re-estimate the parameters of the error process. If β were known we could, of course, use it and the whole model could be returned to a pure autoregressive model. The following Theorem states that using the estimated residuals $y_t - \hat{\beta}x_t$ results in the least squares estimators of $\phi(B)$ having exactly the same asymptotic distributions as the ones obtained from the true residuals, that is, from pure autoregressive models.

Theorem 2.2. Let $\hat{\varepsilon}_t = y_t - \hat{\beta}x_t$, where $\hat{\beta}$ is the OLS estimate of β as defined in Theorem 2.1. Then ordinary least squares regression in $\phi(B)\hat{\varepsilon}_t = \text{error}_t$ results in the estimators, $\hat{\phi}_i$, of the coefficients of $\phi(B)$, which have the same asymptotic distributions as the estimators obtained from the regression of $\phi(B)\varepsilon_t = a_t$, that is, using the true errors in the estimation.

Proof. See the Appendix C.

According to Theorem 2.2. if the errors have an AR(1) model with $\phi_1 = 1$, then $n(\hat{\phi}_1-1)$ has the asymptotic distribution of Dickey & Fuller (1979) and Evans & Savin (1981). If the errors have an AR(2) model with $|\phi_1| < 2$ and $\phi_2 = -1$, then $n(\hat{\phi}_2+1)$ has the asymptotic distribution of Ahtola & Tiao (1985a). Specifically, $n(\hat{\phi}_1-1)$ and $n(\hat{\phi}_2+1)$can be used to test for the existence of a unit root and a pair of complex roots on the unit circle, respectively. We may note that in the latter case the asymptotic distribution of $n(\hat{\phi}_2+1)$ does not depend on ϕ_1, the parameter which determines the periodicity of the error process.

3. Extensions

Three straightforward extensions of the above procedure are immediate.

First the cases where the error process has some other purely non-stationary autoregressive process. This case poses no difficulty as long as we know the asymptotics of the OLS estimates of the co-efficients of this autoregressive process. The procedure goes as earlier.

Secondly the case of more than one explanatory variable is easily solved if for instance the vector $\underset{\sim}{x}_t$ of k explanatory variables is jointly stationary with each individual x_{jt} having an absolutely summable autocovariance function. In this case we can merely think of β as a k-vector and the same with x_t in all the results of the paper. Of course the asymptotic variance of β in Theorem 2.1. becomes a covariance matrix.

The third extension is the case, where the error process is of a mixed nonstationary-stationary type. So let us assume that the autoregressive operator is of the form $\phi(B)\alpha(B)$, where the roots of $\alpha(B)$ are outside the unit circle, and $\phi(B)$ has nonmultiplicative roots on the unit circle.

It can be shown that OLS in $\phi(B)y_t$ = error_t results in the estimators $\tilde{\phi}_i$, such that $\tilde{\phi}_i = \phi_i + O_p(n^{-1})$. Therefore, we have as fast a convergence as in Lemma 2.1. for this initial estimate. The proof of the above result in purely autoregressive models can be found in Ahtola & Tiao (1985b).

At the next stage we should estimate β and the parameters of $\alpha(B)$ <u>jointly</u> from the regression model with autoregressive errors using $\hat{y}_t$'s and $\hat{x}_t$'s, as defined in Theorem 2.1., as our data. Any asymptotically efficient (relative to ML) estimation method could be used at this stage to obtain $\hat{\beta}$ and $\hat{\alpha}$ and the well known results of the asymptotic distributions for $\sqrt{n}(\hat{\beta}-\beta)$ and $\sqrt{n}(\hat{\underset{\sim}{\alpha}}-\underset{\sim}{\alpha})$ could be inserted into Theorem 2.1.

At the final stage we would form $\hat{\varepsilon}_t = \hat{\alpha}(B)(y_t-\hat{\beta}x_t)$ and proceed as in Theorem 2.2. to estimate $\phi(B)$. Again, we obtain asymptotic results, which coincide with the known distributional results of the purely nonstationary autoregressive process.

References

Ahtola, J. and Tiao, G.C. (1985a). "Distributions of least squares estimators of autoregressive parameters for a process with complex roots on the unit circle". J. Time Series Analysis, to appear.

Ahtola, J. and Tiao, G.C. (1985b). "A note on asymptotic inference in autoregressive models with roots on the unit circle". J. Time Series Analysis, to appear.

Dickey, D.A. and Fuller, W.A. (1979). "Distribution of the estimators for autoregressive time series with a unit root." J. Amer. Statist. Assoc., 74, 427-431.

Evans, G.B.A. and Savin, N.E. (1981). "The calculation of the limiting distribution of the least squares estimator of the parameter in a random walk model." Ann. Statist., 9, 1114-1118.

Tiao, G.C. and Tsay, R.S. (1983). "Consistency properties of least squares estimates of autoregressive parameters in ARMA models." Ann. Statist., 11, 856-871.

Tsay, R.S. (1984). "Regression models with time series errors." J. Amer. Statist. Assoc., 79, 118-124.

Appendix A

Proof of Lemma 2.1: For shortness we only prove the case of $\phi(B) = 1 - B$

$$y_t = \phi_1 y_{t-1} + u_t, \tag{A1}$$

where $\phi_1 = 1$ and

$$u_t = \beta(x_t - x_{t-1}) + a_t.$$

OLS in (A1) gives

$$\tilde{\phi}_1 - 1 = \frac{\Sigma y_{t-1}a_t + \beta\Sigma y_{t-1}(x_t - x_{t-1})}{\Sigma y_{t-1}^2} \tag{A2}$$

Unless otherwise noted the summations are from 1 to n. Now

$$\Sigma y_{t-1}^2 = 0_p(n^2)$$

$$\Sigma y_{t-1}a_t = \beta\Sigma x_{t-1}a_t + \Sigma\varepsilon_{t-1}a_t = 0_p(n)$$

$$\Sigma y_{t-1}(x_t - x_{t-1}) = \beta\Sigma x_t(x_t - x_{t-1}) + \Sigma\varepsilon_{t-1}(x_t - x_{t-1}).$$

Since x_t is stationary, $\Sigma x_t(x_t - x_{t-1}) = 0_p(n)$. Therefore to prove the Lemma we need to show that $\Sigma\varepsilon_{t-1}(x_t - x_{t-1}) = 0_p(n)$. (Note that the use of Cauchy-Schwartz would only give $\Sigma\varepsilon_{t-1}(x_t - x_{t-1}) = 0_p(n^{3/2})$, which would yield $\tilde{\phi}_1 - 1 = 0_p(n^{-1/2})$.) Without loss of generality we may assume $E(x_t) = 0$, and without effecting the asymptotics we may set $x_o = 0$, $a_o = 0$. Now

$$\Sigma\varepsilon_{t-1}(x_t - x_{t-1}) = \Sigma\varepsilon_{t-1}x_t - \Sigma\varepsilon_{t-1}x_{t-1}$$

and

$$\Sigma\varepsilon_{t-1}x_t = a_1 \sum_{j=2}^{n} x_j + a_2 \sum_{j=3}^{n} x_j + \ldots + a_{n-1}x_n.$$

$$E(\Sigma\varepsilon_{t-1}x_t) = 0$$

$$E(\Sigma\varepsilon_{t-1}x_t)^2 = \sigma^2\{E(\sum_{j=2}^{n} x_j)^2 + E(\sum_{j=3}^{n} x_j)^2 + \ldots + E(x_n)^2\}$$

Now $E(\sum_{j=i}^{n} x_j)^2 = \sum_{j=i}^{n}\sum_{k=i}^{n} \gamma(j-k)$, where $\gamma(h)$ is the autocovariance of x_t at lag h.

Therefore,

$$E(\Sigma\varepsilon_{t-1}x_t)^2 = \sigma^2 \sum_{i=2}^{n}\sum_{j=i}^{n}\sum_{k=i}^{n} \gamma(j-k)$$

$$= \sigma^2 \sum_{i=2}^{n}\sum_{h=-(n-i)}^{n-i} (n - |h|)\gamma(h).$$

For any fixed i, $\sum_{h=-(n-i)}^{n-i} (n - |h|)\gamma(h) = O(n)$, thus $E(\Sigma\varepsilon_{t-1}x_t)^2 = O(n^2)$, which now implies $\Sigma\varepsilon_{t-1}x_t = O_p(n)$. Similarly $\Sigma\varepsilon_{t-1}x_{t-1} = O_p(n)$. Therefore $\Sigma\varepsilon_{t-1}(x_t - x_{t-1}) = O_p(n)$. Using the established order results in (A2) now gives $\tilde{\phi}_1 - 1 = O_p(n^{-1})$, Q.E.D.

Appendix B

Proof of Theorem 2.1: We, again, prove this only for $\phi(B) = 1 - B$.

$$y_t - \tilde{\phi}_1 y_{t-1} = \beta(x_t - \tilde{\phi}_1 x_{t-1}) + \varepsilon_t - \tilde{\phi}_1 \varepsilon_{t-1} \tag{B1}$$

OLS in (B1) gives

$$\sqrt{n}(\hat{\beta} - \beta) = \frac{\frac{1}{\sqrt{n}} \Sigma (x_t - \tilde{\phi}_1 x_{t-1})(\varepsilon_t - \tilde{\phi}_1 \varepsilon_{t-1})}{\frac{1}{n} \Sigma (x_t - \tilde{\phi}_1 x_{t-1})^2} \tag{B2}$$

The stochastic convergence, to a constant γ^2, of the denominator is clear, since $x_t - \tilde{\phi}_1 x_{t-1}$ is stationary. In the numerator

$$\varepsilon_t - \tilde{\phi}_1 \varepsilon_{t-1} = a_t - (\tilde{\phi}_1 - 1)\varepsilon_{t-1}$$

Threfore, using the results of Lemma 2.1

$$\frac{1}{\sqrt{n}} \Sigma (x_t - \tilde{\phi}_1 x_{t-1})(\varepsilon_t - \tilde{\phi}_1 \varepsilon_{t-1}) = \frac{1}{\sqrt{n}} \Sigma (x_t - \tilde{\phi}_1 x_{t-1}) a_t + 0_p(n^{-1/2})$$

Inserting this into (B2), we immediately see that

$$\sqrt{n}(\hat{\beta} - \beta) \overset{\mathcal{L}}{\to} N(0, \sigma^2 \gamma^{-2})$$

where $\gamma^2 = \text{plim} \frac{1}{n} \Sigma (x_t - \tilde{\phi}_1 x_{t-1})^2$ Q.E.D.

Appendix C

Proof of Theorem 2.2: We, again, prove this only for $\phi(B) = 1 - B$.

$$y_t - \hat{\beta}x_t = \phi_1(y_{t-1} - \hat{\beta}x_t) + v_t \tag{C1}$$

where $\phi_1 = 1$ and $v_t = a_t + (\hat{\beta} - \beta)(x_t - x_{t-1})$.

OLS in (C1) gives

$$\hat{\phi}_1 - 1 = \frac{\Sigma(y_{t-1} - \hat{\beta}x_{t-1})a_t + (\hat{\beta} - \beta)\Sigma(y_{t-1} - \hat{\beta}x_{t-1})(x_t - x_{t-1})}{\Sigma(y_{t-1} - \hat{\beta}x_{t-1})^2} \tag{C2}$$

Adding and subtracting βx_{t-1} inside each of the three summations in (C2) and using the results in Lemma 2.1 and Theorem 2.1 it is straightforward to see that

$$n(\hat{\phi}_1 - 1) = \frac{\frac{1}{n}\Sigma\varepsilon_{t-1}a_t}{\frac{1}{n^2}\Sigma\varepsilon_{t-1}^2} + O_p(n^{-1/2}) \tag{C3}$$

The first term on the right hand side of (C3) comes from applying OLS to $\varepsilon_t = \phi_1\varepsilon_{t-1} + a_t$ and obtaining $n(\hat{\phi}_1 - 1)$, Q.E.D.

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