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## ON THE SENSITIVITY OF THE SOLUTION OF A LINEAR ECONOMETRIC MODEL

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# ON THE SENSITIVITY OF THE SOLUTION OF A LINEAR ECONOMETRIC MODEL

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## 1. INTRODUCTION

This paper deals with the sensitivity of the reduced-form coefficients and especially the solution of a linear econometric model when the structural-form coefficients of the model are allowed to vary according to some special rules. The starting point is an existing econometric model the inaccuracy of which is expressed by specifying the values of its key structural-form coefficients as closed intervals. All the other coefficients are kept fixed at some initial values and so are all the predetermined variables as well. A model specified in this manner generates a set of feasible solutions of its endogenous variables. The focus of interest lies on the geometric properties of the solution set, especially on certain plane or line projections of the set, corresponding to interesting pairs or individuals of endogenous variables, respectively.

The line of thought adopted here differs considerably from the usual treatment of uncertainty aspects in econometric modelling. We do not start from statistical theory in order to construct confidence ellipsoids to our structural-form parameters but we express our view of the uncertainty of the model´s coefficients in a rougher, perhaps more subjective way. This is a deliberate choice: we think that in practice, especially in connection with models designed primarily for forecasting purposes, it may be well justified to be content with somewhat "loose" uncertainty expressions such as described here. The use of most sophisticated statistical means may not be necessary if the model structure in question has ingredients which in fact are of subjective quality.

Neither has our approach very much in common with the sensitivity considerations discussed in the theory of linear equation systems. For example, in the literarure of numerical analysis much attention has been paid in connection with ill-conditioned matrices to represent a scalar measure, or a "condition number", to describe the propensity of the matrix inverse to change due to small variations in the matrix to be inverted. Although our problems are basically not so far from those discussions our approach is different and we make no straight use of those results.

The benefits of our outcomes can be immediately seen: the approach we have used, sometimes called "geometric approach", gives us an outlook concerning the properties of the model solution. If the model is used as a tool in forecasting the picture tells us useful things concerning the reliability of our forecast. It is equally important that our calculations improve our understanding of the model in question. The projection results indicate us indirectly the "dangerous" areas in parameter space, i.e. they warn us of apparent singularity directions where the model can explode as the matrix of the endogenous variables becomes singular.

In the general case we don´t make any special restrictions to the number of equations, the coefficients of which are allowed to vary. In this paper, however, we confine ourselves mainly to the case where only one equation may have varying coefficients the rest being kept constant. This restriction helps us to concentrate on some very basic sub-problems of the general case, the understanding of which is essential when we try to catch the general problem . Also the fundamental geometric characteristics of the general case will be clearly outlined already in the case of one varying equation.

In the following use is made of the so called pivotal operations. Thus after the formulation of the problem one section is devoted to a concise introduction of pivotal operations (for a more comprehensive description, see e.g. Väliaho, 1970 and 1979). That introduction is succeeded by a section where the case of one varying equation will be gone through rather circumstancially. In the last section the links between the case of only one varying equation and the general case are discussed and a conjectural solution strategy is designed for the general case.

## 2. FORMULATION OF THE PROBLEM

We write the structural form (SF) of our econometric model as

$$(1) \qquad \Gamma y = Bz + d$$

where $\Gamma$ is the $(n \cdot n)$ coefficient matrix of the endogenous variables´ vector y, B is the $(n \cdot k)$ matrix of the predeterminated variables´ vector z and d is the vector of the residual terms of the model. In this study we are not interested in the dynamic properties of the model. Thus all lagged endogenous variables are thought to be placed in the predeterminated variables´ vector z and no time subscript is needed in our notations.

Provided that the matrix $\Gamma$ is non-singular the reduced form (RF) of the model can be written as

$$(2) \qquad y = \Pi z + Cd\ ,$$

where we have denoted $\Pi = \Gamma^{-1}B$ and $C = \Gamma^{-1}$.

Now the general formulation of the problem proceeds as follows: the vectors z and d are assumed to be fixed at some initial values, $z = z^0$ and $d = d^0$. The inaccuracy of the model coefficients is taken into account by specifying a set of closed intervals to the parameters in the following way:

$$(3) \qquad \underline{\Gamma} \leq \Gamma \leq \bar{\Gamma}\ ,\ \underline{B} \leq B \leq \bar{B}\ .$$

These sets are called "parallelotopes" in the parameter space. The upper and lower bounds to matrices $\Gamma$ and B are constructed on the ground of all economic, statistical and judgemental information which the model user may have in a particular, say forecasting, situation. In practice a great majority of the model's coefficients may be fixed (their upper and lower bounds coinciding) only a small fraction of the coefficients being really varying (their lower bounds being smaller than their upper bounds).

The matrix $\Gamma$ is assumed to be regular throughout the area (3). However, our procedure will reveal immediately if this assumption is violated. For practical reasons we will also assume that the equations of the model are normalized, that is the diagonal elements of $\Gamma$ are and remain at unity.

What we are now interested in are the geometric properties of the solution set

$$(4) \qquad \mathcal{Y} = \{y \mid \Gamma y = Bz^0 + d^0 , \ \underline{\Gamma} \leq \Gamma \leq \bar{\Gamma} , \ \underline{B} \leq B \leq \bar{B}\}$$

which we call a "polytope" in accordance with the terminology of Ritschard & Rossier (1981). Especially we try to find the *projections* of the set $\mathcal{Y}$ on any selected *line* corresponding to an interesting endogenous variable, say $y_f$, and on any selected *plane*, say $(y_f, y_g)$, corresponding to an interesting pair of endogenous variables.

We will see that the crucial step is then to solve the problems

$$(5) \qquad \begin{cases} \max\ (\min)\ y_f \\ \text{subject to} \\ y \in \mathcal{Y} \end{cases}$$

An important subproblem of (5) is generated in the case when only one equation is allowed to vary.

Let us now fix the matrices $\Gamma$ and B at some initial values, $\Gamma = \Gamma^0$ and $B = B^0$. The equations of the model can now be written in the component form[1)]

$$(6) \qquad \Gamma^0_{i.}y = B^0_{i.}z^0 + d^0_i \ , \ i = 1,\dots,n \ .$$

We call the solution of (6) the *basic solution* of the model and denote it with $y^0$:

$$(7) \qquad y^0 = \Pi^0 z^0 + C^0 d^0 = (\Gamma^0)^{-1}(B^0 z^0 + d^0) \ .$$

Next we allow one of the equations, say the r:th equation, to vary so that

$$(8) \qquad \begin{cases} \underline{\Gamma}_{r.} \leq \Gamma_{r.} \leq \bar{\Gamma}_{r.} \\ \\ \underline{B}_{r.} \leq B_{r.} \leq \bar{B}_{r.} \end{cases}$$

All the other equations remain unaltered. We denote the number of varying coefficients in the r:th equation with $K_r$, that is, $K_r$ is the number of

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1) For the sake of notational convenience we make use of row and column partitions of matrices. For example, for $\Gamma$ we write

$$\Gamma = \begin{bmatrix} \Gamma_{1.} \\ \vdots \\ \Gamma_{n.} \end{bmatrix} = [\Gamma_{.1} \ \cdots \ \Gamma_{.n}] \ .$$

those $\gamma_{rj}$ and $\beta_{rk}$ coefficients, the upper and lower bounds of which do not coincide.

Corresponding to $K_r$ varying parameters there are obviously $2^{K_r}$ *extreme points* in parameter space, i.e. points where every varying coefficient is set at one of its extreme values (either at the upper bound or the lower bound). It is not difficult to prove that in the case of a regular (non-singular) polytope any endogenous variable $y_f$ takes its maximum (minimum) value at an extreme point of the parameter space. Thus it is sufficient to go through the $2^{K_r}$ extreme points in order to find extreme values of an arbitrary element of the vector y. The applying of such an enumeration strategy is not necessary, however, but the problem can be solved in an efficient way.

Consider again the initial situation (6). We now move the coefficients of the r:th equation into any values chosen arbitrarily from the area (8) so that

$$(9) \quad \begin{cases} \Gamma_{r\cdot}^{(1)} = \Gamma_{r\cdot}^{0} + \Delta\Gamma_{r\cdot} \\ B_{r\cdot}^{(1)} = B_{r\cdot}^{0} + \Delta B_{r\cdot} \end{cases}$$

Corresponding to this choice of coefficient values there is a unique model solution which we denote with $y^{(1)}$. Now it can be shown (see Appendix 1) that exactly the same solution $y^{(1)}$ could be attained as well by not moving the coefficients from their initial values but instead by changing the residual term of the r:th equation in a specific way: by giving the residual term $d_r$ a new value

(10) $$d_r^{(1)} = d_r^0 + \delta_r ,$$

where the correction term $\delta_r$ is

(11) $$\delta_r = \frac{-\Delta\Gamma_{r.}y^0 + \Delta B_{r.}z^0}{1 + \Delta\Gamma_{r.}C^0_{.r}}$$

and keeping the rest of the model in its initial state the same model solution $y^{(1)}$ will be generated.

This shows us that any acceptable (non-singular) choice of coefficient values of the r:th equation leads to a model solution which differs from the basic solution by a vector which is constant up to scalar multiplication. The difference vector is simply the r:th column of the original inverse matrix $C^0$ multiplied by the appropriate scalar $\delta_r$.

Thus in this case the polytope is nothing but a line segment in $\mathbb{R}^n$,

(12) $$\Delta y = y^{(1)} - y^0 = \delta_r C^0_{.r} .$$

For example, on the $(y_f, y_g)$ plane the polytype is projected on the line which goes through the point $(y_f^0, y_g^0)$ and the slope of which is $c^0_{gr}/c^0_{fr}$, $c^0_{gr}$ and $c^0_{fr}$ being elements of the vector $C^0_{.r}$.

Thus in the case of one varying equation our problem of finding the boundary of a plane projection will be reduced to the problem of finding the end points of a certain line segment. Those end points are, of course, associated with the extreme values of the correction term $\delta_r$, subject to restrictions (8). Hence we can formulate our problem as follows: solve the optimization problems

$$
(13) \quad \begin{cases} \text{Maximize, minimize} \\ \delta_r(\Delta\Gamma_{r.}\,, \Delta B_{r.}) \\ \text{subject to} \\ \underline{\Gamma}_{r.} - \Gamma^0_{r.} \leq \Delta\Gamma_{r.} \leq \bar{\Gamma}_{r.} - \Gamma^0_{r.} \\ \underline{B}_{r.} - B^0_{r.} \leq \Delta B_{r.} \leq \bar{B}_{r.} - B^0_{r.}\,, \end{cases}
$$

where $\delta_r$ is as defined in (11).

It turns out that the problems (13) can be reformulated as two *linear fractional programming* (LFP) problems with exceptionally simple constraints.

## 3. THE PIVOTAL OPERATION

Following the treatment of Väliaho (1979) but using different symbols we write a linear equation system of n equations in m unknowns

(14) $u = Av$

in the form of a *table* as follows

(15) $$A : u = \begin{array}{c} v \\ \boxed{A} \end{array} .$$

Now we perform in (15) the variable exchange $u_r \longleftrightarrow v_s$, i.e., we solve the r:th equation for $v_s$ and substitute the solution into all the other equations. This leads to a new table

(16) $$A^+ : u^+ = \begin{array}{c} v^+ \\ \boxed{A^+} \end{array} ,$$

which is obtained from (15) by interchanging the places of $u_r$ and $v_s$ and replacing the matrix A by the matrix $A^+ = [a^+_{ij}]$, where

(17) $$\begin{cases} a^+_{rs} = 1/a_{rs} \\ a^+_{is} = a_{is}/a_{rs} , \quad i \neq r \\ a^+_{rj} = -a_{rj}/a_{rs} , \quad j \neq s \\ a^+_{ij} = a_{ij} - a_{is}a_{rj}/a_{rs} , \quad i \neq r, \ j \neq s . \end{cases}$$

We say that the variable exchange $u_r \longleftrightarrow v_s$ as well as the corresponding matrix transformation (17) is carried by a *single pivotal operation* and we denote it by $P_{rs}$. The operation is defined provided that the corresponding *pivot element* $a_{rs} \neq 0$.

We write the formula (17) shortly as $A^+ = P_{rs}(A)$ and adopt the convention $P_{hk}P_{rs}(A) = P_{hk}(P_{rs}(A))$. We say that two pivotal operations $P_{hk}$ and $P_{rs}$ are *dependent* if their pivot elements are either on the same row or on the same column. Otherwise, including the case $(h,k) = (r,s)$, the operations are said to be *independent*.

It is easy to see that two independent pivotal operations commute, i.e.

(18) $$P_{hk}P_{rs}(A) = P_{rs}P_{hk}(A) , \quad r \neq h , \; s \neq k .$$

Furthermore we see that the pivotal operation is involutory, i.e.

(19) $$P_{rs}P_{rs}(A) = I(A) = A ,$$

where $I$ denotes the identity operation.

For two dependent pivotal operations we have a set of simplifying rules from which we shall need the following (for the proof, see Väliaho, 1970):

(20) $$P_{hs}P_{rs}(A) = C^{(R)}_{rh}P_{hs}(A) = P_{rs}C^{(R)}_{rh}(A) ,$$

where $C^{(R)}_{rh}$ denotes a permutation under which the rows r and h of the operand matrix (table) are interchanged, all the other rows remaining unaltered.

A generalization of the single pivotal operation is the *block pivotal operation*. Here we have the operand matrix A being partitioned as $A = [A_{ij}]$. Let now $A_{rs}$ be a non-singular square matrix. The block pivotal operation $P_{(rs)}$ with $A_{rs}$ as the *pivot matrix* takes the form $P_{(rs)}(A) = A^+ = [A^+_{ij}]$, where

$$(21) \qquad \begin{cases} A^+_{rs} = A^{-1}_{rs} \\ A^+_{is} = A_{is}A^{-1}_{rs} \;, \; i \neq r \\ A^+_{rj} = -A^{-1}_{rs}A_{rj} \;, \; s \neq j \\ A^+_{ij} = A_{ij} - A_{is}A^{-1}_{rs}A_{rj} \;, \; i \neq r \;, \; s \neq j \;. \end{cases}$$

The block pivotal operation can be constructed as a product of appropriate single pivotal operations. The interpretation of the block pivotal operation in terms of variable exchanges is straightforward.

Next we want to express the transformation of our basic model from the structural form (1) to the reduced form (2) by means of pivotal operations. In order to do that we write the SF into a table as follows

$$(22) \qquad (\Gamma\, B\, g) : d = \begin{array}{c} \begin{array}{ccc} y & -z & 1 \end{array} \\ \boxed{\begin{array}{ccc} \Gamma & B & g \end{array}} \end{array}$$

The last column in the table is an auxiliary column consisting of vector g which is the value of the "exogenous part" of the model,

$$(23) \qquad g = Bz^0 + d^0 ,$$

depending on the relevant value of B. Thus we read the table (22) as $d = \Gamma y - Bz$.

We need a way to refer shortly to the rows and columns of the table (22). In order to do so we define three index sets $N = \{1,\dots,n\}$, $M = \{n+1,\dots,n+k\}$ and $L = \{n+k+1\}$. Obviously, by N we will first refer to the elements of y and d, by M to those of (minus) z and by L to the auxiliary column of the table.

Choose now $\Gamma$ as the pivot matrix in (22) and apply the block pivotal operation $P_{NN}$. As a result we get a new table (24) which corresponds to the RF of the model:

$$(24) \qquad (\Gamma^+ \; B^+ \; g^+) \; : \; y = \begin{array}{ccc} d & -z & 1 \\ \hline \multicolumn{1}{|c}{C} & -\Pi & \multicolumn{1}{c|}{-y^0} \\ \hline \end{array}$$

The auxiliary column of the table now consists of the basic solution vector $y^0$ (with reversed sign) which thus proves to be the counterpart of $g^0$ on the reduced form side of the model.

If we now applied the operation $P_{NN}$ once again, C as the pivot matrix, we would arrive back at the table (22). Moving from (22) to (24), or vice versa, means a vectoral variable exchange d $\longleftrightarrow$ y. Needless to say, between the SF(22) and RF(24) there is a number of mixed forms, corresponding to variable exchanges between some but not all elements of d and y. We call those forms partially reduced, or *semi-reduced forms* (SRF). An arbitrary SRF can be achieved by means of an appropriately chosen product of single pivotal operations, the choice depending on the starting point table (SF, RF or any other SRF).

Now we want to represent a procedure, with which we can update an existing reduced form table when the corresponding structural form has been changed.

First we construct an augmented table where we have both the original SF table ($\Gamma^0\ B^0 g^0$) and the new SF table ($\Gamma^*\ B^*\ g^*$), where $g^0 = B^0 z^0 + d^0$ and $g^* = B^* z^0 + d^0$. We concentrate on the table elements by writing the table simply as

$$(24)\qquad \begin{array}{c|ccc|} & N & M & L \\ \hline N & \Gamma^0 & B^0 & g^0 \\ N^* & \Gamma^* & B^* & g^* \\ \hline \end{array}\ ,$$

where the index set $N^* = \{n+1,\ldots,2n\}$. Performing now $P_{NN}$ yields the table

$$(25)\qquad \begin{array}{c|ccc|} & N & M & L \\ \hline N & C^0 & -\Pi^0 & -y^0 \\ N^* & \Gamma^* C^0 & B^* - \Gamma^* \Pi^0 & g^* - \Gamma^* y^0 \\ \hline \end{array}$$

or, equivalently,

$$(26)\qquad \begin{array}{c|ccc|} & N & M & L \\ \hline N & C^0 & -\Pi^0 & -y^0 \\ N^* & I+\Delta\Gamma C^0 & \Delta B - \Delta\Gamma\Pi^0 & -\Delta\Gamma y^0 + \Delta B z^0 \\ \hline \end{array}$$

where $\Delta\Gamma = \Gamma^* - \Gamma^0$ and $\Delta B = B^* - B^0$. As we see, we have the basic RF table on the first n rows. The last n rows refer to the new model. By performing now $P_{N^*N}$ the table (26) will be transformed to

$$(27)\qquad \begin{array}{c|ccc|} & N & M & L \\ \hline N & C^* & -\Pi^* & -y^* \\ N^* & \Gamma^0 C^* & B^0 - \Gamma^0 \Pi^* & \delta \\ \hline \end{array}\ ,$$

where

(28.1) $\delta = g^0 - \Gamma^0 y^*$ .

or, equivalently,

(28.2) $\delta = (I + \Delta\Gamma C^0)^{-1}(-\Delta\Gamma y^0 + \Delta B z^0)$

The result (27) can be immediately seen by applying the block form generalization of the property (20),

(29) $$P_{N^*N}P_{NN}(A) = C^{(R)}_{NN^*}P_{N^*N}(A) .$$

The formula (28.2) for the vector $\delta$ is a vectoral generalization of the formula (11) for the scalar correction term $\delta_r$.

In the sequel we will be interested in the relative change of the determinant of the model, expressed as the ratio $|C^*|/|C^0|$ which we will denote with $\rho$. Because we have $C^* = C^0(I + \Delta\Gamma C^0)^{-1}$ we can write the expression for $\rho$ as

(30) $$\rho = |C^*|/|C^0| = |(I + \Delta\Gamma C^0)^{-1}|$$
$$= 1/|\Gamma^* C^0| .$$

In the case of one varying equation, say the r:th, the formulae above will be simplified considerably. Instead of the table (24) we construct a SF table with only one new row,

$$
(31) \qquad \begin{array}{c|ccc|}
 & N & M & L \\
\hline
N & \Gamma^0 & B^0 & g^0 \\
(n+1) & \Gamma^*_{r.} & B^*_{r.} & g^*_r \\
\hline
\end{array} ,
$$

where $g^*_r = B^*_{r.}z^0 + d^0_r$. Performing $P_{NN}$ leads now to the RF table

$$
(32) \qquad \begin{array}{c|ccc|}
 & N & M & L \\
\hline
N & C^0 & -\Pi^0 & -y^0 \\
(n+1) & \iota'_r + \Delta\Gamma_{r.}C^0 & \Delta B_{r.} - \Delta\Gamma_{r.}\Pi^0 & -\Delta\Gamma_{r.}y^0 + \Delta B_{r.}z^0 \\
\hline
\end{array}
$$

$\iota_r$ being a n-vector with 1 as the r:th element and zeros elsewhere.

In order to update the RF table we perform now $P_{n+1,r}$ which gives us the table (33) with new RF matrices on the first n rows,

$$
(33) \qquad \begin{array}{c|ccc|}
 & N & M & L \\
\hline
N & C^* & -\Pi^* & -y^* \\
(n+1) & \Gamma^0_{r.}C^* & B^0_{r.} - \Gamma^0_{r.}\Pi^* & \delta_r \\
\hline
\end{array} .
$$

The correction term is

$$
(34) \qquad \delta_r = g^0_r - \Gamma^0_{r.}y^*
$$

$$
= (1 + \Delta\Gamma_{r.}C^0_{.r})^{-1}(-\Delta\Gamma_{r.}y^0 + \Delta B_{r.}z^0)
$$

cf. formula (11). The first term in the product formula of $\delta_r$ is in fact the value of $\rho$ in the case of one varying equation,

$$
(35) \qquad \rho_r = (1 + \Delta\Gamma_{r.}C^0_{.r})^{-1} .
$$

## 4. SOLUTION TO THE PROBLEM OF ONE VARYING EQUATION

We will consider the set

$$(36)\qquad \mathcal{Y} = \{y \mid \Gamma y = B^0 z^0 + d^0 \,,\ \underline{\Gamma}_{r.} \leq \Gamma_{r.} \leq \bar{\Gamma}_{r.}\} \subset \mathbb{R}^n$$

and point out that in this particular case the set $\mathcal{Y}$ is a line segment in the space $\mathbb{R}^n$. In order to simplify the presentation we assume first that matrix B is fixed, $B = B^0$. The algorithm described here solves the endpoints of this line segment.

We start by solving the following LFP-problems (see (13)),

$$(37.1)\qquad \bar{\delta}_r = \max \delta_r \,(\Gamma_{r.} \mid \underline{\Gamma}_{r.} \leq \Gamma_{r.} \leq \bar{\Gamma}_{r.})$$

and

$$(37.2)\qquad \underline{\delta}_r = \min \delta_r \,(\Gamma_{r.} \mid \underline{\Gamma}_{r.} \leq \Gamma_{r.} \leq \bar{\Gamma}_{r.})$$

after which we are able to update the basic model. We denote the endpoints of $\mathcal{Y}$ with

$$\bar{y} = y^0 + C^0_{.r}\bar{\delta}_r$$

and

$$\underline{y} = y^0 + C^0_{.r}\underline{\delta}_r \,,$$

respectively.

We assume a given SF-choice $\Gamma^0, g^0$ and its RF-side counterparts $C^0$ and $y^0$. Without any loss of generality we assume that the parallellotope is of the form

$$\Gamma^0_{r.} \leq \Gamma_{r.} \leq \bar{\Gamma}_{r.} \quad .$$

The maximizing problem is then in a standard LFP format,

$$(38) \quad \begin{cases} \max \ \delta_r = \dfrac{-\Delta\Gamma_{r.} y^0}{1+\Delta\Gamma_{r.} C^0_{.r}} \\ \text{subject to} \\ \Delta\Gamma_{r.} \leq \nabla_r \\ \Delta\Gamma_{r.} \geq 0 \end{cases}$$

where we have written

$$\nabla_r = \bar{\Gamma}_{r.} - \Gamma^0_{r.} \quad \text{(the "width" of the parallellotope).}$$

In order to control potential singularities we must check that the demoninator in (38), i.e. the change of the determinant

$$\rho_r^{-1} = 1 + \Delta\Gamma_{r.} C^0_{.r}$$

does not change its sign. Therefore we add to the constraints a requirement $\rho_r > 0$ and we have the problem

$$(39)\quad \begin{cases} \max\ \delta_r = \dfrac{-\Delta\Gamma_{r.}y^0}{1+\Delta\Gamma_{r.}C^0_{.r}} = -\rho_r\Delta\Gamma_{r.}y^0. \\ \text{subject to} \\ \Delta\Gamma_{r.} \leq \nabla \\ \rho_r > 0 \\ \Delta\Gamma_{r.} \geq 0 \end{cases}$$

We transform the program to an equivalent linear one by defining a new variable, namely

$$(40)\qquad w_r' \equiv \rho\ \Delta\Gamma_{r.}$$

Now we can write the original maximizing problem (39) as a minimizing problem[1)]

$$(41)\quad \begin{cases} \max\ -\delta_r = (\rho_r\ w_r')\begin{pmatrix} 0 \\ y^0 \end{pmatrix} \\ \text{subject to} \\ (\rho_r\ w_r')\begin{pmatrix} 1 & \nabla_r \\ C^0_{.r} & -I \end{pmatrix} \geqq (1\ \ \vec{0}) \\ \rho_r > 0\ ,\ w_r' \geq 0\ . \end{cases}$$

The LP problem (41) is formulated in terms of row vectors and their post-multiplications. The ordinary Simplex procedure, described in e.g. Väliaho (1976), which makes use of pivotal operations defined above, assumes a column vector and premultiplication matrix layout. In order to avoid transposing of the system (41) we define a slightly modified pivotal operation $P^*_{rs}$ as follows: $P^*_{rs}(A) = A^*$, where

1) In fact, by definition $\rho_r + w_r'\,C^0_{.r} = 1$, so that the first restriction should be a strict equality. However, our procedure will lead to identical results.

$$
(42) \quad \begin{cases} a^*_{rs} = 1/a_{rs} \\ a^*_{is} = -a_{is}/a_{rs} \; , \; i \neq r \\ a^*_{rj} = a_{rj}/a_{rs} \; , \; j \neq s \\ a^*_{ij} = a_{ij} - a_{is}a_{rj}/a_{rs} \; , \; i \neq r, \;\; j \neq s \; . \end{cases}
$$

By using $P^*_{rs}$ instead of $P_{rs}$ we may express a solving procedure for a problem like (41) in a table context without first transforming the problem to its transpose. The procedure is summarized in Appendix 2.

Now we set the problem (41) into a table as follows:

(43)

| | $-\delta_r$ | $S_0$ | $S_N$ | |
|---|---|---|---|---|
| 1 | 0 | -1 | $\vec{0}$ | (-1) |
| $\rho_r$ | 0 | 1 | $\nabla_r$ | (0) |
| $w_r$ | $y^0$ | $c^0_{\cdot r}$ | $-I$ | (N) |
| | (-1) | (0) | (N) | |

Note that the row and column indexation (in brackets) starts from -1 in the table above. Besides that we have in (43) another set of symbols for the rows and columns of the table; from them $S_0$ and $S_N = \{S_1,\ldots,S_n\}$ refer, naturally, to the restrictions of the LP problem.

Solving the problem (41), table (43) as a starting point, means in practice performing an appropriate series of pivotal operations until the operand table has been transformed to one which is feasible, i.e. the very first row being non-negative, and dual feasible, i.e. the very first column being non-negative. The general criteria for choosing the next operation in each stage can be found from Appendix 2.

However, it turns out that in solving (41) we manage with considerably less effort than in the general LP problem case. This follows from the facts that the feasibility of the table can be immediately achieved by one single pivotal operation and that once the feasibility has been reached then in searching for dual feasibility only diagonal pivotal operations are needed.

We recall that the first restriction of the problem should be realized as a strict equality, $\rho_r + w_r' C^0_{.r} = 1$. Thus we must in any case "switch on" this identity. We start by performing the corresponding operation $P^*_{00}$ and we obtain a new table

$$(44)\qquad \begin{array}{c|ccc|c} & -\delta_r & \rho_r & s_N & \\ \hline 1 & 0 & 1 & \nabla_r & (-1) \\ s_0 & 0 & 1 & \nabla_r & (0) \\ w_r & y^0 & -c^0_{.r} & -(I+c^0_{.r}\nabla_r) & (N) \\ \hline & (-1) & (0) & (N) & \end{array}$$

We see immediately that the table (44) is feasible ($a_{-1,j} \geq 0$ for every j).

Next we show, by induction, that only diagonal pivotations are involved in searching for dual feasibility for table (44). Assume first a feasible table (44). The first step in selecting the pivotal element $a_{\nu\mu}$ now leads to the choice of $\nu$ such that $a_{\nu,-1} = \min \{a_{i,-1} \mid i \in N \cup \{0\}\}$. Assuming that $a_{\nu,-1} < 0$, i.e. that at least one element of y is negative, leads then to the comparison of the ratios

$$\text{(i)} \qquad -\frac{\nabla_{r,\nu}}{1 + c^0_{\nu r}\nabla_{r,\nu}} \qquad\qquad (\text{for } \mu = \nu)$$

and

(ii) $\quad -\frac{1}{c^0_{\nu r}}$ $\qquad$ (for all $\mu \neq \nu$) ,

the former of which is greater when $c^0_{\nu r} > 0$. If $c^0_{\nu r} \leq 0$ no comparisons are needed because in that case the off-diagonal candidates are directly disqualified. Thus in the first step a diagonal choice is made.

Secondly, assume that k pivotal operations have been performed, all of them diagonal. We show that the next pivotal operation will be diagonal as well. We denote the initial table (43) with $A^{(0)}$ and the current table with $A^{(H)}$. We denote with the index set $H = \{h_1, \ldots, h_k\}$, $H \subset N$, those diagonal pivotal operations which already have taken place:

$$A^{(H)} = (\prod_{i \in H} P^*_{ii}) P^*_{00} A^{(0)}.$$

Note that $H \subset N$, i.e. we assume that our first restriction has not been "switched off" thus far.

We make a partition of N into two mutually exclusive subsets H and $\bar{H}$. The initial table, corresponding to this partition, can be written

$$(45) \qquad A^{(0)}: \quad \begin{array}{c|cccc|} & -\delta_r & S_0 & S_H & S_{\bar{H}} \\ \hline 1 & 0 & -1 & \vec{0}_H & \vec{0}_{\bar{H}} \\ \rho_r & 0 & 1 & \nabla_{rH} & \nabla_{r\bar{H}} \\ w_{rH} & y^0_H & c^0_{Hr} & -I_H & 0 \\ w_{r\bar{H}} & y^0_{\bar{H}} & c^0_{\bar{H}r} & 0 & -I_{\bar{H}} \\ \hline \end{array}$$

when first an appropriate series of row and column permutations has taken place. The sub-indexation of the elements in the table above should be self-explaining.

Using the commutativity of $P^*$-operations,

$$(\prod_{i\in H} P^*_{ii})P^*_{00} = P^*_{00}(\prod_{i\in H} P^*_{ii})$$

it is easy to verify that the current table takes the form

$$(46) \qquad A^{(H)}: \quad \begin{array}{c|cccc|} & -\delta_r & \rho_r & w_{rH} & s_{\bar{H}} \\ \hline 1 & \delta^H_r & \rho^H_r & \rho^H_r\nabla_{rH} & \rho^H_r\nabla_{r\bar{H}} \\ s_0 & \delta^H_r & \rho^H_r & \rho^H_r\nabla_{rH} & \rho^H_r\nabla_{r\bar{H}} \\ s_H & -y^0_H + \delta^H_r C^0_{Hr} & \rho^H_r C^0_{Hr} & -I_H + \rho^H_r C^0_{Hr}\nabla_{rH} & \rho^H_r C^0_{Hr}\nabla_{r\bar{H}} \\ w_{r\bar{H}} & y^0_{\bar{H}} - \delta^H_r C^0_{\bar{H}r} & -\rho^H_r C^0_{\bar{H}r} & -\rho^H_r C^0_{\bar{H}r}\nabla_{rH} & -I_{\bar{H}} - \rho^H_r C^0_{\bar{H}r}\nabla_{r\bar{H}} \\ \hline \end{array}$$

where

$$\rho^H_r = (1 - \nabla_{rH}C^0_{Hr})^{-1}$$

and

$$\delta^H_r = \rho^H_r(-\nabla_{rH}y^0_H) .$$

If the column (-1) is non-negative we have reached dual feasibility. If not, we choose $\nu$ such that $a_{\nu,-1} = \min\{a_{i,-1} \mid i \in N \cup \{0\}\}$ and face the following comparisons:

1) The case $\nu = 0$. We see that $\max\{a_{-1,i}/a_{0,i} \mid a_{0,i} < 0\}$ is not defined. This means that the objective function is not bounded from below. The explanation for this is that matrix $\Gamma$ is not regular everywhere in the feasible region $\underline{\Gamma} \leq \Gamma \leq \bar{\Gamma}$. Thus our procedure has revealed the singularity of $\Gamma$ and we terminate our search here.

2) The case $\nu \in H$: compare the ratios

(i) $$\frac{\rho_r^H \nabla_{r\nu}}{-1 + \rho_r^H c_{\nu r}^0 \nabla_{r\nu}} \quad \text{(for all } \mu = \nu\text{)}$$

and

(ii) $$\frac{1}{c_{\nu r}^0} \quad \text{(for all } \mu \neq \nu\text{)},$$

the former of which is greater when $c_{\nu r}^0 < 0$. If $c_{\nu r}^0 \geq 0$ the off-diagonal candidates are immediately disqualified and no comparisons are needed[1].

3) The case $\nu \in \bar{H}$: compare the ratios

(i) $$\frac{\rho_r^H \nabla_{r\nu}}{-(1 + \rho_r^H c_{\nu r}^0 \nabla_{r\nu})} \quad \text{(for } \mu = \nu\text{)}$$

and

(ii) $$-\frac{1}{c_{\nu r}^0} \quad \text{(for } \mu \neq \nu\text{)},$$

the former of which is greater if $c_{\nu r}^0 > 0$. If $c_{\nu r}^0 \leq 0$ the off-diagonal candidates will again be ruled out from the game.

We see that in both cases (2-3) a diagonal choice $\mu = \nu$ will be made, provided that $\rho_r$ is bounded everywhere in the feasible region of $\Gamma$. This completes the proof.

We can now summarize our procedure for solving the problem (41) as follows:

---

1) Note also that $\rho_r^H$ in fact bears the current value of $\rho_r$ and is thus necessarily non-negative.

A. Starting from table (43), perform $P^*_{00}$ in order to achieve a feasible table (44).

B. Determine $\nu$ from $a_{\nu,-1} = \min\{a_{i,-1} \mid i \in N \cup \{0\}\}$

C. If $a_{\nu,-1} \geq 0$ go to End 1

If $a_{\nu,-1} < 0$ and $\nu = 0$ go to End 2

If $a_{\nu,-1} < 0$ and $\nu \in N$ go to D

D. Perform $P^*_{\nu\nu}$ and go to B

End 1: The solution has been found.

End 2: The objective function is not bounded from below.

We note that each performance of D has a very clear interpretation: When $P^*_{\nu\nu}$ takes place it means switching the value of $\gamma_{r\nu}$ from the lower bound to the upper bound or vice versa. Thus our procedure also verifies the result that in the case of a regular polytope any endogenous variable takes its maximum (minimum) value at some extreme point of the parameter space (cf. page 7).

Now we have constructed a scheme for the search of $\bar{\delta}_r$. The corresponding minimizing problem can be solved analogously. Once we know $\bar{\delta}_r$ and $\underline{\delta}_r$ we can update the model and get $\bar{y}$ and $\underline{y}$, respectively.

In defining the set y in (36) it was assumed that the β-coefficients were fixed, $B = B^0$. The assumption was made for the sake of convenience only. The procedure above can be quite easily modified to include the case where also the r:th row of B is allowed to vary. Anyhow, because that generalization is nothing but a mechanical exercise we do not consider it here.

## 5. ON THE SOLUTION OF THE GENERAL CASE

In this chapter we turn to the case where several equations are allowed to vary simultaneously. We concentrate again on the consequences of variations in $\Gamma$ and fix the matrix B at $B = B^0$. Our polytope is thus

(47) $$\mathcal{Y} = \{ y \mid \Gamma y = g^0 , \underline{\Gamma} \leq \Gamma \leq \bar{\Gamma} \}$$

where, as before, $g^0 = B^0 z^0 + d^0$ .

Before we can proceed in searching any plane projections of $\mathcal{Y}$ we must be able to solve the line projection problems

(48) $$\begin{cases} \max\ (\min)\ y_f \\ \text{subject to} \\ y \in \mathcal{Y} \end{cases}$$

The problem (48) is not trivial. In order to solve it we propose a heuristic procedure which seems to work in all convex cases and in some non-convex ones as well.

Let us again start from some initial setting $\Gamma = \Gamma^0$. We still assume that $\Gamma$ is regular throughout its feasible region. Under this assumption we know that the maximum and minimum values of $y_f$ will be found at some extreme points of the parameter space. Therefore we will probably save work by not choosing $\Gamma^0$ quite freely from the feasible area but by setting it equal to some extreme point of the parameter space, $\Gamma^0 = \underline{\Gamma}$, for example.

We have denoted the number of varying coefficients in the i:th equation with $K_i$. Many equations of the model may have fixed coefficients and they are not very interesting from our point of view. In order to separate varying equations from fixed equations we define an index set $N_v \subseteq N$ ,

$$N_v = \{ i \mid K_i > 0,\ i \in N \} .$$

We denote the number of elements in $N_v$ with $n_v$.

Starting from the chosen $(\Gamma^0, g^0)$ pair and its RF-counterpart $(C^0, y^0)$ we now consider the equations $i \in N_v$, one at a time. For every equation i in question we set the problems

$$(49) \quad \begin{cases} \max\,(\min)\; y_f \\ \text{subject to} \\ y \in \{y \mid \Gamma y = g^0 ,\ \underline{\Gamma}_{i.} \leq \Gamma_{i.} \leq \bar{\Gamma}_{i.} ,\ \Gamma_{j.} = \Gamma^0_{j.} ,\ j \neq i \} \end{cases}$$

Now we can make use of the LFP-scheme described in the preceding chapter. As a result we get a set of maximum and minimum values of $\delta_i$, i.e. $\bar{\delta}_i$ and $\underline{\delta}_i$, $i \in N_v$. We denote the set with $\Delta_v$.

If we are dealing with the maximizing problem we next select the element of $\Delta_v$ which corresponds to the fastest increase in the value of $y_f$. The comparison is based on the fact that the selection of $\delta_i$ would give $y_f$

a new value

$$y_f = y_f^0 + c_{fi}^0 \delta_i .$$

We are thus dealing with a simple maximizing problem

$$(50) \quad \begin{cases} \max\ c_{fi}^0 \delta_i \\ \text{subject to} \\ \delta_i \in \Delta_v \end{cases}$$

Let the solution of (50) be $\hat{\delta}_j$. If it now turns out that the choice of $\hat{\delta}_j$ would not increase $y_f$ the maximum has already been found ($y_f^0$).

If, however, the value of $y_f$ can be increased we update the j:th equation of the model (i.e. the row $\Gamma_{j.}$) corresponding to the coefficient values associated with $\hat{\delta}_j$. We now get a new SF ($\Gamma^1, g^0$) and calculate the corresponding new RF($C^1, y^1$), cf. page 16. Next we rename the current model (superscript 1) to the initial stage (superscript 0) and go back to resolve the problem (49). The process is repeated until the value of $y_f$ can no longer be increased. Then we have attained a local optimum which in the convex case is the global optimum as well.

If we are lucky enough to choose a good extreme point as our initial stage $\Gamma^0$ we will save much effort. A very simple "switching" procedure may help us here. It is well known that the partial derivative $\frac{\partial y_i}{\partial \gamma_{rs}}$ in a linear model $\Gamma y = g^0$ takes the form

$$(51) \quad \frac{\partial y_i}{\partial \gamma_{rs}} = -c_{ir} y_s .$$

We can make use of this simply by going through all the varying cofficients of the model and switching every one of them either on its upper or its lower bound, depending on the sign of $\frac{\partial y_f}{\partial \gamma_{rs}} = -c^0_{fr}y^0_s$ . Of course, we must already have *some* initial RF $(C^0,y^0)$ available, corresponding e.g. for $\Gamma^0 = \underline{\Gamma}$. Every single "switching away" from $\Gamma^0$ will change both C and y and there is no guarantee that the signs of the elements in C and y would at that time stay unchanged. Nevertheless, it is likely that this extremely simple procedure will give us a reasonably good starting point. In fact, if the partial derivatives happen to be sign-constant for every feasible $\Gamma$ (as may very well be the case in some practical applications) the simple switching procedure will lead us straight to the optimum.

Once we have reached the maximum value of $y_f$ we update matrix $\Gamma$ correspondingly. We denote the solution with $\tilde{y}$ and the corresponding $\Gamma$-matrix with $\tilde{\Gamma}$. Clearly, we are now located not only on the edge of the $(y_f,y_g)$-projection of $\mathcal{Y}$ but also in some extreme point (corner) of the boundary. If we now studied the $n_v$ varying equations one by one, solving the max(min)$\delta_i$-problems for every $i \in N_v$, $\tilde{\Gamma}$ as reference point, we would see that in every case either max $\delta_i$ or min $\delta_i$ is equal to zero. We denote the non-zero max/min $\delta_i$ with $\tilde{\delta}_i$.

Corresponding to each varying equation $i \in N_v$ there is on the $(y_f,y_g)$ plane a directed line segment from $(\tilde{y}_f,\tilde{y}_g)$ to $(\tilde{y}_f^{(i)},\tilde{y}_g^{(i)})$, where

$$\tilde{y}_j^{(i)} = \tilde{y}_j + \tilde{c}_{ji}\tilde{\delta}_i \quad , \; j = f,g$$

Clearly, every model solution subject to a choice of $\Gamma_{i.}$ from the feasible area $\underline{\Gamma}_{i.} \leq \Gamma_{i.} \leq \bar{\Gamma}_{i.}$ , $\Gamma_j = \tilde{\Gamma}_j$ , $j \neq i$, would be projected on the line segment in question.

We want to move along the border of the projection without visiting any interior points. Let us choose the counter-clockwise direction. We need a reference direction in order to be able to select right the next varying equation. A good choice for the first reference direction is the direction of the $y_g$-axis. We measure the angle between the reference direction and the directed line segment corresponding to the i:th varying equation and denote the result with $\theta_i$, $i \in N_v$. Because we start from $(\tilde{y}_f, \tilde{y}_g)$, where $\tilde{y}_f = \max y_f$, $y \in \mathcal{Y}$, we know that $0 < \theta_i < \frac{\pi}{2}$, $i \in N_v$. Let $\theta_s = \min_{i \in N_v} \theta_i$. This tells us that the next varying equation will be the s:th equation.

In order to jump to the next corner of the projection we need the solution of the LFP problem of the preceding chapter, having $\underline{\Gamma}_{s.} \leq \Gamma_{s.} \leq \bar{\Gamma}_{s.}$ and the rest of the coefficients fixed at $\tilde{\Gamma}$. We tune the s:th equation corresponding to the non-zero solution $(\tilde{\delta}_s)$ of the max(min) $\delta_s$-problems and obtain the new SF and RF coordinates, $\tilde{\tilde{\Gamma}}$, $\tilde{\tilde{C}} = (\tilde{\tilde{\Gamma}})^{-1}$ and $\tilde{\tilde{y}}$, say.

The next phase is again to solve LFP problems

$$\max(\min)\ \delta_j\ (\Gamma_{j.} | \underline{\Gamma}_{j.} \leq \Gamma_{j.} \leq \bar{\Gamma}_{j.}\ ,\ \Gamma_{i.} = \tilde{\tilde{\Gamma}}_{i.}\ ,\ i \neq j)\ ,\ j \in N_v \setminus \{s\}$$

Solving the problems for $j = s$ is not required because we already know the answer.

If for every j either max $\delta_j$ or min $\delta_j$ is zero we are still in an extreme point of the boundary. We assume that this is the case. We denote the non-zero solutions of max/min $\delta_j$ with $\tilde{\tilde{\delta}}_j$, $j \in N_v \setminus \{s\}$. We choose now the direction from $(\tilde{y}_f, \tilde{y}_g)$ to $(\tilde{\tilde{y}}_f, \tilde{\tilde{y}}_g)$ as the reference direction and consider the $n_v - 1$ directed line segments from $(\tilde{\tilde{y}}_f, \tilde{\tilde{y}}_g)$ to $(\tilde{\tilde{y}}_f^{(i)}, \tilde{\tilde{y}}_g^{(i)})$, where

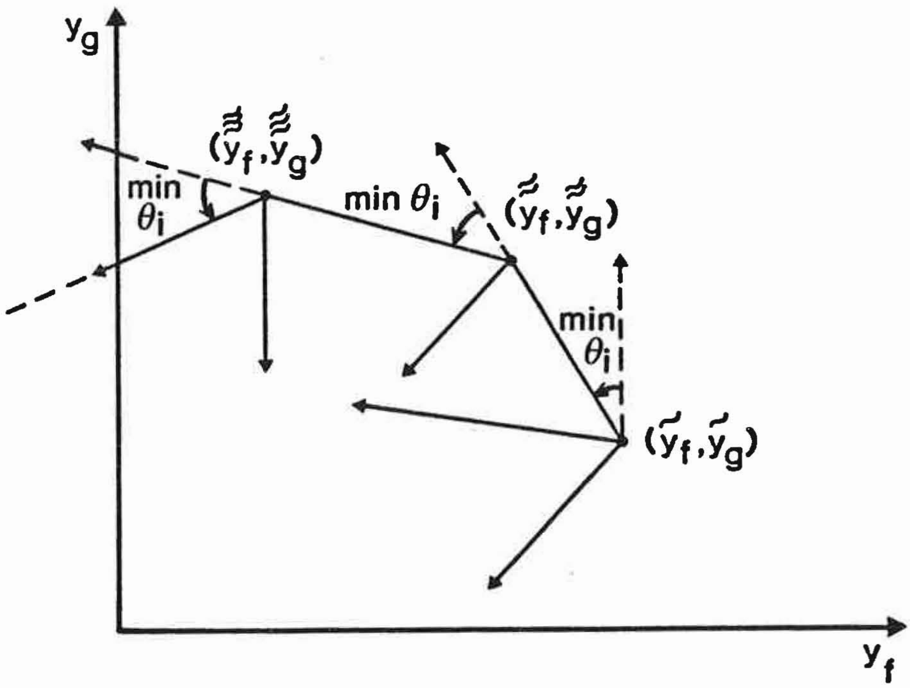
$$\approx{y}_j^{(i)} = \approx{y}_j + \approx{c}_{ji}\approx{\delta}_i \ , \ j = f,g \ , \ i \in N_v \setminus \{s\} \ .$$

As before, we denote the angle between the reference direction and the directed line segment corresponding to the i:th varying equation with $\theta_i$. Because our projection can generally be non-convex, some $\theta_i$ may be negative and we can only say that $-\frac{\pi}{2} < \theta_i < \frac{\pi}{2}$. What we are interested in is to find $\min\limits_{i \in N_v \setminus \{s\}} \theta_i$ and to select the next varying equation corresponding to it.

Once we have chosen the next varying equation, the u:th equation for example, we proceed in the same way as described above, i.e. tuning the equation according to $\approx{\delta}_u$ and thus obtaining the new SF and RF coordinates ($\approx{\Gamma}$, $\approx{C}$ and $\approx{y}$), then solving the relevant max(min) $\delta_j$-problems ($j \in N_v \setminus \{u\}$), checking if either max $\delta_j$ or min $\delta_j$ is zero for all j and so on. We always take as the reference direction the edge which we have just travelled along and base the selection of the next varying equation on the angles $\theta_i$. Proceeding this way will carry us - we suggest - along the boundary of the projection, jumping over unnecessary boundary points and never getting lost inside the projection, finally arriving at the departure point $(\tilde{y}_f, \tilde{y}_g)$.

Figure 1 illustrates the first phases of the procedure.

Figure 1: Illustration of the corner-hunting procedure. The case of three varying equations.




Our heuristic procedure could be further simplified if we knew in advance that the projection in question will be convex. In that case the selection of the next varying equation can be based on comparing the slopes of the line projections, without need to solve explicitly the other endpoints of the line projections. On the other hand, the check that either max $\delta_j$ or min $\delta_j$ is still zero in every corner would be lost.

Finally, let us have one more look at the basic problem (48). An interesting special case arises when the set $\mathcal{Y}$ is convex in the following manner.

Let us start from some $\Gamma^0$, this time arbitrarily chosen from the feasible area of $\Gamma$. Now we solve the LFP-problems

$$(52)\quad \begin{cases} \max\ (\min)\ \delta_i \\ \text{subject to} \\ \underline{\Gamma}_{i\cdot} \leq \Gamma_{i\cdot} \leq \bar{\Gamma}_{i\cdot} \\ \Gamma_{j\cdot} = \Gamma^0_{j\cdot}\ , \quad j \neq i \end{cases}$$

for all $i \in N_v$. We denote the $\Gamma_{i\cdot}$ row vector which maximizes (minimizes) $\delta_i$ with $\tilde{\Gamma}^0_{i\cdot}$ ($\dot{\Gamma}^0_{i\cdot}$), $i \in N_v$. For the rows $i \in N \setminus N_v$ we have $\tilde{\Gamma}^0_{i\cdot} = \dot{\Gamma}^0_{i\cdot} = \Gamma^0_{i\cdot}$. We collect the rows $\tilde{\Gamma}^0_{i\cdot}$ into the matrix $\tilde{\Gamma}^0$ and the rows $\dot{\Gamma}^0_{i\cdot}$ into $\dot{\Gamma}^0$.

Consider now the convex set

$$(53)\qquad \mathcal{J}(y^0) = \{y \mid \tilde{\Gamma}^0 y \leq g^0\ ,\ \dot{\Gamma}^0 y \geq g^0\}$$

It is rather easy to show that

$$(54)\qquad \mathcal{J}(y^0) \subseteq \mathcal{Y}$$

where $\mathcal{Y}$ stands for the original polytope (47). The relation (54) holds true regardless of the choice $\Gamma^0$.

On the other hand, consider the model

$$(55)\quad \begin{cases} \tilde{\Gamma}^0_{i.}y = g^0_i \\ \Gamma^0_{j.}y = g^0_j \quad , \quad j \neq i \quad . \end{cases}$$

We write (55) in matrix form as $\tilde{\Gamma}^{i,0}y = g^0$ and we denote its solution with $\tilde{y}^i$. We have, of course

$$(56)\quad \begin{cases} \Gamma^0_{i.}\tilde{y}^i = g^0_i + \tilde{\delta}_i \\ \Gamma^0_{j.}\tilde{y}^i = g^0_j \quad , \quad j \neq i \end{cases}$$

where $\tilde{\delta}_i$ now refers to the maximal value of $\delta_i$ in (52). Naturally, $\tilde{\delta}_i \geq 0$.

Let us now orientate back to $y^0 = (\Gamma^0)^{-1}g^0$ from $\tilde{y}^i$ by means of the $\delta_i$ term. We denote the appropriate correction term with $\tilde{\delta}^*_i$ and we can write

$$(57)\quad \begin{cases} \tilde{\Gamma}^0_{i.}y^0 = g^0_i + \tilde{\delta}^*_i \\ \Gamma^0_{j.}y^0 = g^0_j \quad , \quad j \neq i \quad . \end{cases}$$

It can be shown that

$$(58)\qquad \tilde{\delta}^*_i = -\,(1 + (\tilde{\Gamma}^0_{i.} - \Gamma^0_{i.})C^0_{.i})\tilde{\delta}_i$$

where $C^0_{.i}$ is the i:th column of $C^0 = (\Gamma^0)^{-1}$. Now, one of our basic assuptions is that (see page 18)

$$1 + (\tilde{\Gamma}^0_{i.} - \Gamma^0_{i.})C^0_{.i} > 0$$

and thus we have $\tilde{\delta}^*_i \leq 0$.

Because we have

(59) $$y^0 = \tilde{y}^i + \tilde{\delta}^*_i \tilde{C}^{i,0}_{.i}$$

where $\tilde{C}^{i,0} = (\tilde{\Gamma}^{i,0})^{-1}$, we obtain, by premultiplying (59) by $\tilde{\Gamma}^{i,0}_{i.} = \tilde{\Gamma}^0_{i.}$

(60) $$\tilde{\Gamma}^0_{i.} y^0 = \tilde{\Gamma}^0_{i.} \tilde{y}^i + \tilde{\Gamma}^{i,0}_{i.} \tilde{C}^{i,0}_{.i} \tilde{\delta}^*_i$$

$$= g^0_i + \tilde{\delta}^*_i \leq g^0_i \quad .$$

Analogously it can be shown that

(61) $$\dot{\Gamma}^0_{i.} y^0 \geq g^0_i \quad .$$

By repeating the argument for every equation $i \in N_V$ and gathering the results we obviously have

(62) $$\begin{cases} \tilde{\Gamma}^0 y^0 \leq g^0 \\ \dot{\Gamma}^0 y^0 \geq g^0 \quad . \end{cases}$$

Consider now the special case where solving the problems (52) would lead to the same matrices $\tilde{\Gamma}^0$ and $\dot{\Gamma}^0$ irrespective of the choice $\Gamma^0$. In that fortunate case we would have $\tilde{\Gamma}^0 y \leq g^0$ and $\dot{\Gamma}^0 y \geq g^0$ for every $y \in \mathcal{Y}$, that means $\mathcal{Y} \subseteq \mathcal{S}(y^0)$, and because of (54)

$\mathcal{Y} = \mathcal{S}(y^0)$.

The problems (48) could then be solved as (sign-unconstrained) LP-problems

$$(63) \quad \begin{cases} \max\ (\min)\ y_f \\ \tilde{\Gamma}^0 y \leq g^0 \\ \dot{\Gamma}^0 y \geq g^0 \end{cases} .$$

Thus, in some special cases the line projection problems (48) can be solved as single linear programming problems although the convex set $\mathcal{S}(y^0)$ defined in (53) generally depends on the starting point $\Gamma^0$ and $\mathcal{S}(y^0)$ does not cover the whole set $\mathcal{Y}$. It is even possible that the idea behind (63) can be applied in developing our procedure so that the search of max $y_f$ (or min $y_f$) is arranged as a sequence of LP and LFP problems.

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Appendix 1: Derivation of Formula (11)

Let the initial model be

(A1.1) $\Gamma^0 y = B^0 z^0 + d^0$

with the solution

(A1.2) $y^0 = (\Gamma^0)^{-1}(B^0 z^0 + d^0)$

We now change the coefficients of the row r yielding new coefficient matrices $\Gamma^1$ and $B^1$,

(A1.3) $$\begin{cases} \Gamma^1 = \Gamma^0 + \iota_r \Delta\Gamma_{r.} \\ B^1 = B^0 + \iota_r \Delta B_{r.} \end{cases}$$

where with $\iota_r$ we denote a (column) vector the $r$:th element of which is one the rest being zeros.

We denote the solution of the new model

(A1.4) $\Gamma^1 y = B^1 z^0 + d^0$

with $y^1$,

(A15) $y^1 = (\Gamma^1)^{-1}(B^1 z^0 + d^0)$

Now we show that the same solution $y^1$ can be attained from the model with the initial coefficients as well if the residuals of the model are manipulated in a proper way. In other words, we want to find an additive residual correction vector $\delta$ such that the model

$$\text{(A1.6)} \qquad \Gamma^0 y = B^0 z^0 + d^0 + \delta$$

has the same solution $y^1$ as model (A1.4)

The solution of (A1.6) is

$$\text{(A1.7)} \qquad y = (\Gamma^0)^{-1}(B^0 z^0 + d^0 + \delta) = y^0 + (\Gamma^0)^{-1}\delta$$

which we set equal to $y^1$. On the other hand,

$$\begin{aligned}
y^1 &= (\Gamma^0 + \iota_r \Delta\Gamma_{r.})^{-1}((B^0 + \iota_r \Delta B_{r.})z^0 + d^0) \\
&= (\Gamma^0(I + (\Gamma^0)^{-1}\iota_r\Delta\Gamma_{r.}))^{-1}(B^0 z^0 + d^0 + \iota_r \Delta B_{r.} z^0) \\
&= (I - \tau(\Gamma^0)^{-1}\iota_r\Delta\Gamma_{r.})(\Gamma^0)^{-1}(B^0 z^0 + d^0 + \iota_r \Delta B_{r.} z^0) \\
&= y^0 - \tau(\Gamma^0)^{-1}\iota_r\Delta\Gamma_{r.}y^0 + (I - \tau(\Gamma^0)^{-1}\iota_r\Delta\Gamma_{r.})(\Gamma^0)^{-1}\iota_r\Delta B_{r.}z^0 ,
\end{aligned}$$

where

$$\tau = (1 + \Delta\Gamma_{r.}(\Gamma^0)^{-1}\iota_r)^{-1} .$$

We obtain

$$\begin{aligned}
\delta = \Gamma^0(y^1 - y^0) &= -\tau\iota_r\Delta\Gamma_{r.}y^0 + \Gamma^0(I - \tau(\Gamma^0)^{-1}\iota_r\Delta\Gamma_{r.})(\Gamma^0)^{-1}\iota_r\Delta B_{r.}z^0 , \\
&= -\tau\iota_r\Delta\Gamma_{r.}y^0 + \iota_r\Delta B_{r.}z^0 - \tau\iota_r(\tau^{-1} - 1)\Delta B_{r.}z^0 \\
&= -\tau\iota_r\Delta\Gamma_{r.}y^0 + \tau\iota_r\Delta B_{r.}z^0
\end{aligned}$$

or, in component form,

$$\delta_r = \frac{-\Delta\Gamma_{r.}y^0 + \Delta B_{r.}z^0}{1 + \Delta\Gamma_{r.}(\Gamma^0)^{-1}\iota_r}$$

$$\delta_j = 0 \quad , \quad j \neq r$$

## Appendix 2: Solving a LP-problem in a table context

Väliaho (1976) gives an algorith for solving a LP problem

(A2.1) $\min q = c'x$

subject to

$Ex + f \geq 0$

$x \geq 0$

in a table context, using the pivotal operation $P_{rs}$. In (A2.1), x and c are n-vectors, f is a m-vector and E is a (m x n)-matrix.

Here we reproduce the algorithm, this time having the problem in a transposed form,

(A2.2) $\min q = x'c$

subject to

$v = x'E' + f' \geq 0$

$x' \geq 0$

and using the pivotal operation $P^*_{rs}$.

The algorithm will be started from the table

$$
\text{(A2.3)} \quad B: \quad \begin{array}{c|cc|} & q & v \\ \hline 1 & 0 & f' \\ x & c & E' \\ \hline \end{array}
$$

where the rows are numbered 0,...,n and the columns 0,...,m. We denote $N = \{1,...,n\}$ and $M = \{1,...,m\}$.

Now a set of matrices (tables) B is constructed by a sequence of pivotal operations $P^*_{rs}$, $r \in N$, $s \in M$, starting from (A2.3). At a given stage, the variables attached to the rows $i \in N$ (the nonbasic variables) are given a value of zero whereas the values of the variables attached to the columns $j \in M$ (the basic variables) are obtained from row zero. The element $b_{00}$ yields the current value of the objective function.

A table is *feasible* if $b_{0j} \geq 0$ for all $j \in M$
A table is *dual feasible* if $b_{i0} \geq 0$ for all $i \in N$
A table is *optimal* if it is both feasible and dual feasible.

The algorithm goes as follows:

(A) Start from the table B in (A2.3)

(B) Determine $\lambda$ from $b_{0\lambda} = \min \{b_{0j} | j \in M\}$.
If $b_{0\lambda} \geq 0$ go to D.
If $b_{0\lambda} < 0$ go to C.

(C) Determine $\nu$ from $b_{\nu\lambda} = \max \{b_{i\lambda} | i \in N\}$.
If $b_{\nu\lambda} \leq 0$ go to End 2 .
If $b_{\nu\lambda} > 0$ determine $\mu$ from
$b_{0\mu}/b_{\nu\mu} = \max \{b_{0j}/b_{\nu j} | j \in M\}$ and perform $P^*_{\nu\mu}$ .
If $\mu = \lambda$ go to B .
If $\mu \neq \lambda$ go to C .

(D) Determine $\nu$ from $b_{\nu 0} = \min \{b_{i0} | i \in N\}$ .
If $b_{\nu 0} \geq 0$ go to End 1 .
If $b_{\nu 0} < 0$ determine $\mu$ from
$b_{0\mu}/b_{\nu\mu} = \max \{b_{0j}/b_{\nu j} | j \in M, b_{\nu j} < 0\}$ .

If the maximum is not defined go to End 3 .

Otherwise perform $P^*_{\nu\mu}$ and go to D .

End 1 : The solution has been found.

End 2 : The restrictions are inconsistent.

End 3 : The objective function in not bounded from below in the feasible region.