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ASYMPTOTIC INFERENCE IN AUTOREGRESSIVE MODELS WITH ROOTS ON THE UNIT CIRCLE

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ABSTRACT

An estimation and inference procedure is proposed for parameters of the p^{th} order autoregressive model with roots both on the unit circle and outside the unit circle. The procedure is motivated by the fact that the parameter estimates of the nonstationary part of the model have higher order consistency properties than the parameter estimates of the stationary part. The procedure allows the use of known asymptotic distribution results of purely nonstationary models and purely stationary models. Only ordinary least squares routines are needed.

Keywords: asymptotic theory, autoregressive models, ordinary least squares, roots on the unit circle

1. INTORDUCTION

Consider the p^{th} order autoregressive model

$$\phi(B)(w_t - \mu) = a_t, \quad (1.1)$$

where

$$\phi(B) = U(B)\alpha(B) ,$$

$U(B) = 1 - u_1B - \ldots - u_rB^r$, having all the zeroes on the unit circle

$\alpha(B) = 1 - \alpha_1B - \ldots - \alpha_{p-r}B^{p-r}$, having all the zeroes outside the unit circle

and

$$a_t \sim NID(0,\sigma^2), \qquad t = 1,2,\ldots,n$$

The purpose of this paper is to propose an estimation and inference procedure for the coefficients of $U(B)$ and $\alpha(B)$ such that we can test for the existence of different types of roots on the unit circle in the model (1.1) by a uniform method. A theoretically convenient feature with this procedure is the fact that no new distribution theory is required given that the distribution theory of the ordinary least squares estimates of the purely nonstationary model, where $\alpha(B) = 1$, is known. Dickey (1976), Fuller (1976), Dickey and Fuller (1979), Hasza and Fuller (1979) and Dickey, Hasza and Fuller (1982) treat cases where the zeroes of $U(B)$ are 1 or -1 or where $U(B)$ is the seasonal differencing operator $1-B^s$. Ahtola and Tiao (1984) treat the complex roots case, where $U(B) = 1 - u_1B + B^2$, $|u_1| < 2$.

A practically convenient feature is that only ordinary least squares routines are needed. Another convenience, in particular with the complex roots case is that u_1, which determines the periodicity of the model, need not be fixed when the existence of complex unit roots is tested.

If $U(B)$ has roots equal to 1, μ disappears from (1.1). Otherwise it can be consistently estimated by $\bar{w} = \frac{1}{n} \Sigma w_t$, and $\bar{w}$ substituted for μ without affecting the known asymptotic results, see Dickey (1976) for roots equal to -1 and Ahtola and Tiao (1984) for complex roots on the unit circle. To unify notation we simply write $y_t = w_t - \mu$, thus (1.1) becomes

(1.2) $$\phi(B)y_t = a_t, \qquad t = 1,2,\ldots,n .$$

2. ASYMPTOTIC INFERENCE

Let

(2.1) $$U(B)y_t = b_t,$$

then

(2.2) $$\alpha(B)b_t = a_t .$$

Therefore b_t has a stationary AR(p-r) model. (2.1) can be interpreted as a purely nonstationary AR(r) process for y_t with stationary AR(p-r) disturbances.

On the other hand, define x_t as

(2.3) $$x_t = \alpha(B)y_t$$

then

(2.4) $$U(B)x_t = a_t \; .$$

Therefore x_t has a purely nonstationary AR(r) model and known asymptotic results could be used for the least squares estimators of u_i, $i = 1,\ldots,r$ from (2.4). However, $\alpha(B)$ is unknown and consequently x_t cannot be obtained.

If consistent estimators $\hat{\alpha}_i$ of α_i, $i = 1,\ldots,p-r$ were available, it would be natural to insert these into $\alpha(B)$ to get $\hat{\alpha}(B)$. Then define $\hat{x}_t$ as

(2.5) $$\hat{x}_t = \hat{\alpha}(B)y_t$$

and substitute $\hat{x}_t$ for x_t in (2.4) to estimate U(B). It is shown below that consistent estimation of $\hat{\alpha}(B)$ is possible and that the use of $\hat{x}_t$ in (2.4) results in exactly the same asymptotic distribution theory for the least squares estimators of U(B) as if we knew x_t.

Consistent estimation of $\alpha(B)$ is made possible by the results of Tiao and Tsay (1983), who show that ordinary least squares regression of y_t on $y_{t-1},\ldots,y_{t-r}$ gives consistent estimators for the parameters in U(B). Important in this result is the fact that consistency is fast. Namely, denote by $\tilde{u}_i$ the estimators from this regression, then

(2.6) $$\tilde{u}_i = u_i + O_p(n^{-1}), \qquad i = 1,2,\ldots,r \; .$$

(2.6) makes it possible to estimate $\alpha(B)$ consistently by ordinary least squares from

(2.7) $$\alpha(B)\hat{b}_t = \varepsilon_t \; ,$$

where

$$\hat{b}_t = \tilde{U}(B)y_t \ , \qquad \text{and} \qquad \varepsilon_t = \frac{\tilde{U}(B)}{U(B)} a_t \ .$$

$\hat{\alpha}(B)$ from (2.7) is then substituted for $\alpha(B)$ in (2.3) to get $\hat{x}_t$, which in turn is substituted for x_t in (2.4). Therefore successive application of (2.1) through (2.4) describes the whole procedure. Schematically, we can describe this as

$$(2.1) \longrightarrow \tilde{U}(B), \hat{b}_t; \quad (2.2) \longrightarrow \hat{\alpha}(B); \quad (2.3) \longrightarrow \hat{x}_t; \quad (2.4) \longrightarrow \hat{U}(B) \ .$$

The following theorem summarizes the procedure and its main results.

Theorem 1. Let (1.1) be the true model. Let $\hat{\alpha} = (\hat{\alpha}_1,\ldots,\hat{\alpha}_{p-r})$ be the ordinary least squares estimatros from (2.2), when $\hat{b}_t$ is substituted for b_t, where $\hat{b}_t = (1 - \tilde{u}_1 B - \ldots - \tilde{u}_r B^r)y_t$, and the $\tilde{u}_i$´s are obtained by ordinary least squares from regression of y_t on $y_{t-1},\ldots,y_{t-r}$. Then $\sqrt{n}(\hat{\alpha} - \alpha) \overset{\mathcal{L}}{\to} N(0,\sigma^2\Gamma^{-1})$, where $\Gamma_{ij} = \underset{n\to\infty}{\text{plim}} \frac{1}{n} \sum_t \hat{b}_{t-i}\hat{b}_{t-j}$, $i,j = 1,2,\ldots,p-r$.

Also, let $\hat{x}_t = \hat{\alpha}(B)y_t$. Then ordinary least squares from the regression of $\hat{x}_t$ on $\hat{x}_{t-1},\ldots,\hat{x}_{t-r}$ results in estimators $\hat{u}_1,\ldots,\hat{u}_r$ having the same asymptotic distribution as the ordinary least squares estimators from (2.4) with known x_t´s.

Proof. See the Appendix.

In practice we have found it useful to iterate the procedure by redefining $\hat{b}_t$ as $\hat{b}_t = (1 - \hat{u}_1 B - \ldots - \hat{u}_r B^r)y_t$ and starting from (2.2) with these new $\hat{b}_t$´s. Iteration is stopped when the $\hat{u}_i$´s converge. All the asymptotic results of Theorem 1 remain intact as is shown at the end of the Appendix.

APPENDIX.

Proof of Theorem 1. As will be seen from the proof, the only crucial result we will rely on is (2.6), which holds for any type of roots on the unit circle. Therefore we do not lose any generality by proving the theorem for the complx roots case

$$U(B) = 1 - u_1 B - u_2 B^2, \qquad \text{where } |u_1| < 2, \ u_2 = -1 .$$

Also, for simplicity we assume $p = 3$. The case for general p is a straightforward extension.

The regression of x_t on x_{t-1} and x_{t-2} results in

$$\text{(A1)} \qquad n\begin{bmatrix} \bar{u}_1 - u_1 \\ \bar{u}_2 + 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{n^2}\Sigma x_{t-1}^2 & \frac{1}{n^2}\Sigma x_{t-1}x_{t-2} \\ \frac{1}{n^2}\Sigma x_{t-1}x_{t-2} & \frac{1}{n^2}\Sigma x_{t-2}^2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{n}\Sigma x_{t-1}a_t \\ \frac{1}{n}\Sigma x_{t-2}a_t \end{bmatrix}$$

The regression of $\hat{x}_t$ on $\hat{x}_{t-1}$ and $\hat{x}_{t-2}$ results in

$$n\begin{bmatrix} \hat{u}_1 - u_1 \\ \hat{u}_2 + 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{n^2}\Sigma \hat{x}_{t-1}^2 & \frac{1}{n^2}\Sigma \hat{x}_{t-1}\hat{x}_{t-2} \\ \frac{1}{n^2}\Sigma \hat{x}_{t-1}\hat{x}_{t-2} & \frac{1}{n^2}\Sigma \hat{x}_{t-2}^2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{n}\Sigma \hat{x}_{t-1}\eta_t \\ \frac{1}{n}\Sigma \hat{x}_{t-2}\eta_t \end{bmatrix}$$

Now,

$$\eta_t = \frac{1 - \hat{\alpha}_1 B}{1 - \alpha_1 B} a_t = 1 - \frac{(\hat{\alpha}_1 - \alpha_1)B}{1 - \alpha_1 B} a_t$$

$$= a_t - (\hat{\alpha}_1 - \alpha_1) b_{t-1}$$

and consequently

$$\frac{1}{n}\Sigma\, \hat{x}_{t-i}\eta_t = \frac{1}{n}\Sigma\, \hat{x}_{t-i}a_t - (\hat{\alpha}_1 - \alpha_1)\frac{1}{n}\Sigma\, \hat{x}_{t-i}b_{t-1}, \qquad i = 1,2 .$$

It is shown below that $(\hat{\alpha}_1 - \alpha_1) = O_p(n^{-1/2})$. Also from $\hat{x}_t = (1 - \hat{\alpha}_1 B)y_{t-1}$ and the fact that b_{t-1} is stationary we get

$$\frac{1}{n}\Sigma\, \hat{x}_{t-i}b_{t-1} = O_p(1) , \qquad i = 1,2 .$$

(See Lemma 2.5 in Tiao and Tsay (1983).)

Therefore we have

$$\frac{1}{n}\Sigma\, \hat{x}_{t-i}\eta_t = \frac{1}{n}\Sigma\, \hat{x}_{t-i}a_t + O_p(n^{-1/2})$$

and

$$\text{(A2)} \qquad n\begin{bmatrix} \tilde{u}_1 - u_1 \\ \tilde{u}_2 + 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{n^2}\Sigma\, \hat{x}_{t-1}^2 & \frac{1}{n^2}\Sigma\, \hat{x}_{t-1}\hat{x}_{t-2} \\ \frac{1}{n^2}\Sigma\, \hat{x}_{t-1}\hat{x}_{t-2} & \frac{1}{n^2}\Sigma\, \hat{x}_{t-2}^2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{n}\Sigma\, \hat{x}_{t-1}a_t + O_p(n^{-1/2}) \\ \frac{1}{n}\Sigma\, \hat{x}_{t-2}a_t + O_p(n^{-1/2}) \end{bmatrix}$$

To prove the latter part of the theorem it is sufficient to show that terms in (A1) equal terms in the same position in (A2) plus $O_p(n^{-1/2})$. The method of showing this is exactly the same for each term and the basis of the result lies in the fact that $\tilde{u}_i = u_i + O_p(n^{-1})$ and $\hat{\alpha}_1 = \alpha_1 + O_p(n^{-1/2})$.

As an example we will show that

$$\text{(A3)} \qquad \frac{1}{n^2}\Sigma\, \hat{x}_{t-1}^2 - \frac{1}{n^2}\Sigma\, x_{t-1}^2 = O_p(n^{-1/2}) .$$

Now,

$$\frac{1}{n^2} \Sigma \hat{x}^2_{t-1} - \frac{1}{n^2} \Sigma x^2_{t-1}$$

$$= \frac{1}{n^2} \Sigma \{(1-\hat{\alpha}_1 B) y_{t-1}\}^2 - \frac{1}{n^2} \Sigma \{(1-\alpha_1 B) y_{t-1}\}^2$$

$$= \frac{1}{n^2} \Sigma \{(1-\alpha_1 B) y_{t-1} - (\hat{\alpha}_1 - \alpha_1) y_{t-2}\}^2 - \frac{1}{n^2} \Sigma \{(1-\alpha_1 B) y_{t-1}\}^2$$

$$= -2(\hat{\alpha}_1 - \alpha_1) \frac{1}{n^2} \Sigma (1-\alpha_1 B) y_{t-1} y_{t-2} + (\hat{\alpha}_1 - \alpha_1)^2 \frac{1}{n^2} \Sigma y^2_{t-2} .$$

Since $\frac{1}{n^2} \Sigma y^2_{t-2} = O_p(1)$, (A3) follows if

(A4) $$\hat{\alpha}_1 - \alpha_1 = O_p(n^{-1/2})$$

To verify (A4) recall that

$$\hat{\alpha}_1 = \frac{\frac{1}{n} \Sigma \hat{b}_{t-1} \hat{b}_t}{\frac{1}{n} \Sigma \hat{b}^2_{t-1}} ,$$

where

$$\hat{b}_t = y_t - \tilde{u}_1 y_{t-1} - \tilde{u}_2 y_{t-2}$$

$$= b_t - (\tilde{u}_1 - u_1) y_{t-1} - (\tilde{u}_2 - u_2) y_{t-2} .$$

Since

$$(\tilde{u}_1 - u_1) = O_p(n^{-1}) ,$$

$$(\tilde{u}_2 - u_2) = O_p(n^{-1}) .$$

$$\Sigma\, b_t y_{t-j} = O_p(n)\ , \qquad j = 0,1,2,3$$

and

$$\Sigma\, y_{t-j} y_{t-i} = O_p(n^2)\ , \qquad i,j = 0,1,2,3$$

(see Tiao and Tsay (1983)), we have

$$\frac{1}{n} \Sigma\, \hat{b}_{t-1}\hat{b}_t = \frac{1}{n} \Sigma\, b_{t-1}b_t + O_p(n^{-1})\ .$$

Similarly,

$$\frac{1}{n} \Sigma\, \hat{b}^2_{t-1} = \frac{1}{n} \Sigma\, b^2_{t-1} + O_p(n^{-1})\ .$$

Therefore,

$$\hat{\alpha}_1 = \frac{\frac{1}{n} \Sigma\, b_{t-1}b_t}{\frac{1}{n} \Sigma\, b^2_{t-1}} + O_p(n^{-1})$$

$$= \alpha_1 + \frac{\frac{1}{n} \Sigma\, b_{t-1}a_t}{\frac{1}{n} \Sigma\, b^2_{t-1}} + O_p(n^{-1})\ ,$$

by the definition of b_t.

Thus

(A5) $$\hat{\alpha}_1 - \alpha_1 = \frac{\frac{1}{n} \Sigma\, b_{t-1}a_t}{\frac{1}{n} \Sigma\, b^2_{t-1}} + O_p(n^{-1})\ .$$

Since b_t is stationary, $\frac{1}{n} \Sigma\, b_{t-1}a_t = O_p(n^{-1/2})$ and $\frac{1}{n} \Sigma\, b^2_{t-1} = O_p(1)$, therefore $\hat{\alpha}_1 - \alpha_1 = O_p(n^{-1/2})$.

To prove the first result of the theorem, we see from (A5) that

$$\sqrt{n}\ (\hat{\alpha}_1 - \alpha_1) = \frac{\frac{1}{\sqrt{n}} \Sigma\, b_{t-1}a_t}{\frac{1}{n} \Sigma\, b^2_{t-1}} + O_p(n^{-1/2})$$

Thus

(A6) $$\sqrt{n}\,(\hat{\alpha}_1 - \alpha_1) \overset{\mathcal{L}}{\to} N(0,\sigma^2\Gamma^{-1})$$

where

$$\Gamma = \operatorname*{plim}_{n\to\infty} \frac{1}{n} \Sigma\, b^2_{t-1} .$$

(A6) follows from the asymptotic theory of stationary autoregressive processes.

Γ can be consistently estimated by

$$\hat{\Gamma} = \frac{1}{n} \Sigma\, \hat{b}^2_{t-1} ,$$

which is immediate from the proof of the theorem. Q.E.D.

Iteration

The inclusion of the iteration does not affect the proof above. Since $\hat{u}_i - u_i = O_p(n^{-1})$ at each iteration, we notice that $\hat{\alpha}_1 - \alpha_1 = O_p(n^{-1/2})$ exactly as in (A5) at each iteration. Therefore Theorem 1 applies after any iteration.

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