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RELATIVE CHANGES AND INDEX NUMBERS

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PREFACE

This work has grown out of a chain of statistical problems and their partial solutions met in the short-term forecasting project of the Research Institute of the Finnish Economy. I would like to express my sincere gratitude to the other two original members of the group responsible for the project, Heikki Hämäläinen, Lic.Pol.Sc., and Dr. Pentti Vartia. They have been working on the project ever since it was founded in 1971 by Prof. Meinander, the then director of the Institute. Pentti Vartia has been responsible for the construction of the short-term econometric model of ETLA, of which he has published his doctoral dissertation and Heikki Hämäläinen is preparing his doctoral thesis about the general forecasting framework, which actually produces our biannual forecasts. My role in the project has been that of a statistician.

The construction of a solution program for our econometric model, which uses relative changes of the variables in the behavioural equations, led first to the concept of the logarithmic mean and finally to the new log-change index numbers. This monograph is a continuation of my thesis for licentiate's degree.

Successive directors of the Institute, Prof. Nils Meinander, Prof. Ahti Molander, Dr. Kaarlo Larna and its present director Tauno Ranta, have all provided encouragement, a stimulating atmosphere and good research opportunities which have contributed to the ripening of the study.

I have benefited much from the expert knowledge of Prof. Leo Törnqvist, first in his research seminar at the University of Helsinki Institute of Statistics and afterwards in many valuable discussions with him in his capacity as the scientific adviser to our Institute.

I have had the opportunity to discuss various special problems with my fellow workers and friends. My thanks are due especially to Kari Alho, Jukka Lassila, Pekka Lastikka, Timo Teräsvirta and Pentti Vartia for their interest and the valuable remarks which have improved my presentation considerably.

I have also benefited from the encouragement given by Prof. W.E. Diewert, the University of British Columbia, although our correspondence has merely touched upon the problems dealt with in the present work.

Prof. Eino Haikala and Prof. Seppo Mustonen, as the official examiners of the work, have given me support and valuable advice during the final stages of the study.

The figures were drawn by Miss Ann-Christine Ekebohm and the text (with its endless revisions) was typed by Mrs. Tuula Tamezoujt highly skilfully and with unbelievable tolerance.

My particular thanks are also due to Mr. Jaakko Railo, who has checked my English and translated some chapters of the work.

None of those who have helped me bear any responsibility for the remaining errors and shortcomings of the study.

Finally, it is my pleasant duty to express my thanks to the Yrjö Jahnsson Foundation for the grants awarded to me for this study.

Helsinki, August 1976

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INTRODUCTION

The present study has two chief purposes: to consider indicators of relative changes and apply the ideas arrived at thereby to economic index number calculations.

In chapter 1 the concept of relative change is defined and the properties of various indicators of relative change are examined. The asymmetry of the customary indicator of relative change, $H_1(y/x) = (y-x)/x$, and its poor decomposition properties are well known. Obviously from force of habit, however, this indicator is generally employed even in sophisticated analyses[1)], but part of the explanation may also lie in the fact that the relative change concept has not been considered systematically in the literature and that, in consequence, the alternatives available to this indicator have not been sufficiently well known.

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1) E.g., the trade cycle model of the Dutch Central Planning Bureau and the model of the Research Institute for the Finnish Economy (ETLA), see P. Vartia (1974). For a debate on the possible merits and dismerits of the use of relative changes in econometric models, see P. Vartia (1976 a,b,c), Kukkonen (1975, 1976 a,b), Kanniainen (1976), Teräsvirta (1976) and Teräsvirta and Vartia (1975).

In theoretical considerations and in several sophisticated models the logarithmic change, $H_4(y/x) = \log_e(y/x)$, is used as an indicator of relative change. Törnqvist (1935) p. 36 already stated as his opinion that "there are no good reasons for giving index numbers in the form of percentages, because the natural logarithms of the indices are at least for scientists far more interesting".

Nowadays, when even pocket calculators have keys for natural logarithms and exponentation, the practical difficulties in using log-change as an indicator of relative change have dissappeared. From many a research report, however, the impression is gained that the author has been unable to give this transformation a clear interpretation based on the relative change concept.

The writing of research reports about models involving log-changes is complicated by the absence of simple names for the concepts concerned.

At this point I wish to repeat an earlier suggestion of mine that the log-change $H_4(y/x) = \log_e(y/x)$ multiplied by one hundred - i.e., $100 \log_e(y/x)$ - be called the <u>dynamic change from x to y</u>. The "unit" of this could be called 'dyn' and ordinary change percentages and dyns would then approximately correspond to each other for small changes[1)]. As nit is used as the "unit" for the natural logarithms and hence for the log-change, the numerical values of a relative change expressed in

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1) See Appendix 5 and Herva, Vartia and Vasama (1973). Cf. P. Vartia (1974, p. 33-35), who uses 'log-percent' instead of our 'dyn'. Recently the name 'natural percent' and the symbol <u>%</u> have been proposed by L. Törnqvist.

dyns is 100 times its numerical value expressed in nits. Certain other multiples of the log-change (e.g., 20 $\log_{10}(y/x)$, the unit of which is decibel) have an established position in physics. The dynamic change, as contrasted to the log-change, is suitable especially for the communication of the analysis of economic phenomena, where the relative changes are small as a rule (e.g., + 10 % = 9.531 dyn = 0.09531 nit = 0.8279 db).

Chapter 2 provides a survey of various views about price and volume indices. Special attention is paid to differences between the descriptive (or statistical) approach and the economic theory of index numbers as presented by Samuelson and Swamy (1974). Leontief's (1936) views about index numbers are critically considered and some new results are presented.

Chapter 3 deals with the theory of index numbers associated with customary period analysis. The exposition mainly follows the test approach of Irving Fisher, the aim being to present axiomatically ideas that are usually understood only intuitively.

In chapter 4 we consider "practical" problems inherent in index number construction and various strategies for constructing index series: especially base and chain index methods. The latter topic leads to the subject matter of the next chapter.

The theory of chain index numbers, which was developed by Divisia and Törnqvist independently of each other, is considered in chapter 5. A precise exposition of this theory, based on "continuous time", presents difficulties connected with the definitions of the value, price and quantity concepts that have been attended to hardly at all in any of the publications I have come across.

In chapter 6 new index number formulas already derived in Y. Vartia (1974, 1975) are presented and compared, in terms of the criteria introduced in the preceding chapters, with corresponding previously suggested formulas. My second index, Vartia Index II, was constructed in the beginning of 1974 as the solution to the problem set forth and discussed by Theil (1973). Theil (1973) and Sato (1974) derived good approximative solutions to the problem, which Theil (1974) already regarded as unsolvable. Sato (1975, 1976) reported its independent discovery and proved that our new ideal log-change index is exact for all CES utility (or production) functions.

The two new ideal log-change index numbers prove to be noteworthy competitors of the best known index number's, e.g., Fisher's ideal index, Stuvel's and Törnqvist's indices. Our new indices are especially suitable for chain index calculations.

Empirical index number calculations using the most popular formulas such as Laspeyres' and Paasche's formulas together with precicion formulas of Fisher, Törnqvist, Theil, Stuvel and the author are reported in chapter 7. The data consists of yearly Finnish GDP figures by industries for 1957-72 and monthly figures of imports of fuels and lubricants for January 1972 - September 1974, of which the latter material is especially difficult. From the GDP-material both base and chain indices are calculated and compared. Empirical calculations provide an illustration of the merits and dismerits of various formulas and confirm the more theoretical findings of the work.

1. ON THE ALGEBRAIC THEORY OF RELATIVE CHANGES

Let us consider a positive (or at least non-negative) sequence of numbers (y_t, $t \in Z$) representing the values of some ratio scale variable observed in time.

To the absolute change from the point of time t-1 to the point of time t there corresponds the difference $y_t - y_{t-1}$.

Yet, since the observed values of a ratio scale variable are determinate only up to the unit of measurement (see Vasama and Vartia (1971) p. 49-52), y_t may be replaced as well by $y_t' = ay_t$, where $a > 0$.

The change in the variable, as determined from the sequence (y_t', $t \in Z$) will then be $y_t' - y_{t-1}' = a(y_t - y_{t-1})$.

An essential characteristic of a relative change is that its value is independent of the unit of measurement employed; i.e., the relative change has the same value irrespective of whether it is determined from (y_t) or (y_t').

An <u>indicator of a relative change</u> is defined here as a real valued function C(x,y) defined for all positive x and y, $C: \mathbb{R}^2_+ \to \mathbb{R}$, which has the following characteristic properties A to D:

A. $C(x,y) = 0$ if and only if $x = y$.

B. $C(x,y) > 0$ if and only if $y > x$.

$C(x,y) < 0$ if and only if $y < x$.

Thus the first argument (here x) represents the base value of the variable, the second argument (here y) representing the "new" value of the variable or the value to be compared with the "base value".

C. C is an increasing function of y when x is fixed; and C is continuous.

The properties A to C will be possessed by any function describing change, e.g., the function $C(x,y) = y-x$.

D. $\forall a: a>o \Rightarrow C(ax,ay) = C(x,y)$.

By D any indicator of relative changes will be independent of the unit of measurement.

The following functions C(x,y), for instance, are indicators of relative changes.

1. $(y-x)/x$
2. $(y-x)/y$

3. $(y-x)/\frac{1}{2}(x+y)$

4. $c\log_e(y/x)$ when $c>0$

5. $c(y-x)/\min(x,y)$ when $c>0$

6. $(y-x)/K(x,y)$, where $K(x,y)$ is some <u>mean</u> of x and y.

A mean $K(x,y)$ of two numbers x and y is a real valued function K defined in a region $A \subset \mathbb{R}^2$, which has the following properties.

(1) $\min(x,y) \leq K(x,y) \leq \max(x,y)$.

(2) K is a continuous function.

(3) $\forall a: a>0 \Rightarrow K(ax,ay) = aK(x,y)$.

(4) $K(x,y) = K(y,x)$.

According to this broad definition, the following, for example, are means:

Arithmetic mean	$(x+y)/2$	
Geometric mean	$\sqrt{xy}$	; $x>0$, $y>0$
Harmonic mean	$2/(\frac{1}{x}+\frac{1}{y})$	; $x>0$, $y>0$
Maximum	$\max(x,y)$	
Minimum	$\min(x,y)$	
Moment mean of order k	$[\frac{1}{2}(x^k+y^k)]^{\frac{1}{k}}$	; $x>0$, $y>0$

We define the logarithmic mean $L(x,y)$ of positive numbers x and y as follows:

$$\text{Logarithmic mean } L(x,y) = \begin{cases} (y-x)/\log_e(y/x) & \text{, for } x \neq y \\ x & \text{, for } x = y \end{cases}$$

We have to prove that it really is a mean[1].

Theorem 1: The logarithmic mean $L(x,y)$ satisfies the conditions of a mean (1)-(4) when $x>0$, $y>0$.

Proof: (1) By definition $\log_e(y/x) = (y-x)/L(x,y)$.
This applies to $x = y$, too.

Next use the mean value theorem[2]: for every differentiable function f, there exists a point ξ strictly between x and x+h, $h \neq 0$ such, that

$$f(x+h) - f(x) = f'(\xi)h \quad .$$

Take $f(x) = \log_e(x)$

$$\log_e(y/x) = \log_e(y) - \log_e(x)$$

$$= D\log_e(\xi) \; (y-x)$$

$$= (y-x)/\xi \quad .$$

According to the mean value theorem $\xi = L(x,y)$ is between x and y.

(2) $L(x,y)$ being a ratio of continuous nonzero functions, is a continuous function[3] for every positive and different x and y.

We have to show only that $L(x,y) \to x$, as $x \to y$.

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1) Prof. Seppo Mustonen has generalized in an unpublished paper the logarithmic mean for n positive arguments.

2) See Apostol (1957) p. 93.

3) See Apostol (1957) p. 68.

Using (1) we have $\min(x,y) \leq L(x,y) \leq \max(x,y)$ and therefore $L(x,y) \to x$, as $x \to y$, because both limits approach each other. Thus $L(x,y)$ is a continuous for $x = y$, too.

$$(3)\quad L(ax,ay) = (ay-ax)/\log_e(ay/ax)$$

$$= a\, L(x,y)$$

for every positive a, x and y.

$$(4)\quad L(x,y) = (y-x)/\log_e(y/x)$$

$$= (x-y)/\log_e(x/y)$$

$$= L(y,x) \quad .$$

□

We thus have a very important representation of the log-change

(1) $$\log_e(y/x) = (y-x)/L(x,y) \quad , \; y>0,\; x>0 \quad .$$

This is not just an identity but it says that the log-change is literally a relative change of the form $(y-x)/K(x,y)$, where $K(x,y)$ equals the logarithmic mean $L(x,y)$. Or simply: log-change is a relative change in respect to the logarithmic mean. What is essential is that $L(x,y)$ is a mean.

It can be shown (see appendix 3) that for positive x and y, $x \neq y$

(2) $$\left(\frac{x+y}{2}\right) > L(x,y) > \sqrt{xy} \quad ,$$ and thus

(3) $$\frac{y-x}{\left(\frac{x+y}{2}\right)} \lessgtr \log_e\left(\frac{y}{x}\right) \lessgtr \frac{y-x}{\sqrt{xy}} \quad ,$$ according to if $x<y$ or $x>y$.

All expressions in (2) or (3) are equal if $x = y$.

Theorem 2: Every indicator $C(x,y)$ of the relative change is a function of the ratio y/x alone or, in other words, there is a function $H: \mathbb{R}_+ \to \mathbb{R}$ such that $C(x,y) = H(y/x)$.

Proof: Choosing $a = 1/x$, we have, by D,

$$C(x,y) = C(1,y/x) \triangleq H(y/x). \quad \square$$

The properties A to D, as written for H:

A'. $H(\frac{y}{x}) = 0$ if and only if $\frac{y}{x} = 1$.

B'. $H(\frac{y}{x}) \gtrless 0$ if and only if $\frac{y}{x} \gtrless 1$.

C'. $H: \mathbb{R}_+ \to \mathbb{R}$ is a continuous and increasing function of its argument.

D'. $H(ay/ax) = H(y/x)$.

The best method to compute the ordinary relative change $C(x,y) = (y-x)/x$, for example, is based on the representation $H(y/x) = y/x - 1$.

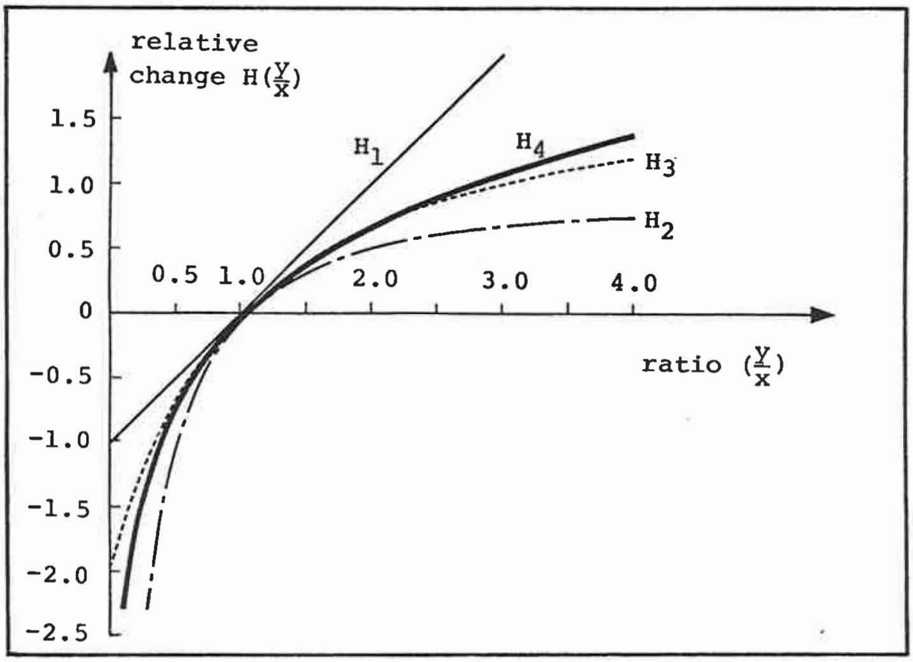
The following important indicators of relative changes are represented graphically in Figure 1:

(4) $$H_1(y/x) = y/x - 1 = (y-x)/x\ ,\quad H_2(y/x) = 1 - x/y = (y-x)/y$$

$$H_3(y/x) = \frac{y/x-1}{\frac{1}{2}(1+y/x)} = \frac{y-x}{\frac{1}{2}(x+y)}\ ,\quad H_4(y/x) = \log_e(y/x)\ .$$

Note the different ranges of these indicators of relative change: $H_1(y/x) \in (-1, \infty)$, $H_2(y/x) \in (-\infty, 1)$, $H_3(y/x) \in (-2, 2)$ and $H_4(y/x) \in (-\infty,$

Figure 1. Selected indicators of relative changes



To exclude the indicators of relative change that do not behave approximately as $(y-x)/x$ does when $y/x \approx 1$, a further requirement will be imposed: the indicators have to be <u>normed</u>, and an indicator H of relative change will be called normed if and only if

E. $$\lim_{S \to 1}\left(\frac{H(S)}{H_1(S)}\right) = \lim_{S \to 1}\left(\frac{H(S)-H(1)}{S-1}\right) = H'(1) = 1 .$$

$H(S) = c\log_e(S)$, for example, will be normed only when $c = 1$. For instance, the indicator of relative changes generally used in electronics

(5) $$db = 20\log_{10}(y/x),$$

expressing the relative change in terms of decibels, is not normed.

Correspondingly, information may be expressed in terms of bits, and then

(6) $$\text{bit} = \log_2(\frac{1}{p}),$$

where $p = P(A)$ is the probability if the event A concerned. This information measure is an indicator of the relative difference between p and the probability 1 of the certain event, which is not normed.

The customarily used indicator of relative change, $H_1(\frac{y}{x}) = (y-x)/x$, is annoyingly asymmetric. For example, the two changes involved in the sequence $x \to y \to x$,

(7) $$H_1(\frac{y}{x}) = \frac{y-x}{x} \quad \text{and} \quad H_1(\frac{x}{y}) = \frac{x-y}{y}$$

are not equal apart of the sign.
On the contrary, if the value of the variable first doubles and then decreases to a half, there will first be an increase of 100 per cent and then a decrease of 50 per cent. Correspondingly, the numerical value of the relative difference between two numbers x and y will change depending on which of them is used as the point of comparison. (In the theory of index numbers this question is considered under the head of the "time reversal test".)

We will define the indicator of relative change, $H(\frac{y}{x})$, as symmetric if and only if

F. $$H(\tfrac{y}{x}) = -H(\tfrac{x}{y}) .$$

For example, the indicators

(8) $$H_1(\tfrac{y}{x}) = \frac{y-x}{x} \quad \text{and} \quad H_2(\tfrac{y}{x}) = \frac{y-x}{y}$$

are asymmetric. By contrast the indicators

(9) $$H_3(\tfrac{y}{x}) = \frac{y-x}{\frac{1}{2}(x+y)} , \quad H_4(\tfrac{y}{x}) = \log_e(\tfrac{y}{x})$$

and, in general, all indicators of type 6, or

(10) $$H(\tfrac{y}{x}) = \frac{y-x}{K(x,y)}$$

are symmetric (and the ones mentioned here are also normed).

One peculiar way of making the indicators (8) symmetric is to define

(11) $$H_5(\tfrac{y}{x}) = \begin{cases} \frac{y-x}{x} , & \text{when } \frac{y}{x} \geq 1 \\ \frac{y-x}{y} , & \text{when } \frac{y}{x} < 1. \end{cases}$$

This is how Rao & Miller advise in their Applied Econometrics (1971) on p. 17, the student to compute "relative changes".

Written in another way, $H_5(\frac{y}{x})$ can be interpreted as the following kind of extreme case of indicators of type 6:

(12) $$H_5(\frac{y}{x}) = \frac{y-x}{\min(x,y)} .$$

Its computational simplicity is the only advantage H_5 possesses in comparison with, for instance, the indicators in (9).
In what follows we shall demonstrate that the indicator

(13) $$H_4(\frac{y}{x}) = \log_e(\frac{y}{x}) ,$$

based on the log-change, is actually the only one recommendable for scientific use[1]. To this end the aggregation and decomposition properties of various indicators will be investigated.

It is interesting to compare the following indicators with one another:

Relative change with respect to the minuend: $$H_1(\frac{y}{x}) = \frac{y-x}{x}$$

Relative change with respect to the arithmetic mean: $$H_3(\frac{y}{x}) = \frac{y-x}{\frac{1}{2}(x+y)}$$

Logarithmic change (the relative change with respect to the logarithmic mean): $$H_4(\frac{y}{x}) = \log_e(\frac{y}{x}) = \frac{y-x}{L(x,y)}$$

1) Some multiples of this indicator are widely used e.g. in electroni and acoustics (desibels, eq. (5)) and in information theory (bits, eq. (6)). Other similar logarithmic 'scales' are the DIN-scale in photography and Richter's scale in seismology measuring the energy of earth quakes.

As mentioned before, all of these three indicators are normed, but H_1 is not symmetric:

$$(14) \qquad H_1(\frac{y}{x}) = \frac{y-x}{x} = -(\frac{y}{x})(\frac{x-y}{y}) = -(\frac{y}{x}) H_1(\frac{x}{y})$$

$$\neq -H_1(\frac{x}{y}) \quad \text{when } \frac{y}{x} \neq 1.$$

In practice, however, H_1 is often dealt with as if it were symmetric when $\frac{y}{x} \approx 1$.

Consider the two-stage change $x \to y \to z$ and examine how the relative change from x to z can be expressed by the various indicators in terms of the changes $x \to y$ and $y \to z$. In other words, we wish to express $H(\frac{z}{x})$ in terms of $H(\frac{y}{x})$ and $H(\frac{z}{y})$.

$$(15) \qquad H_1(\frac{z}{x}) = \frac{z-x}{x} = \frac{(z-y)+(y-x)}{x}$$

$$= \frac{z-y}{x} + \frac{y-x}{x}$$

$$= \frac{y-x}{x} + (\frac{y}{x})(\frac{z-y}{y})$$

$$= H_1(\frac{y}{x}) + (\frac{y}{x}) H_1(\frac{z}{y}) .$$

$$(16) \qquad H_3(\frac{z}{x}) = \frac{z-x}{\frac{1}{2}(x+z)} = \frac{(z-y)+(y-x)}{\frac{1}{2}(x+z)}$$

$$= \frac{y-x}{\frac{1}{2}(x+z)} + \frac{z-y}{\frac{1}{2}(x+z)}$$

$$= (\frac{x+y}{x+z}) H_3(\frac{y}{x}) + (\frac{y+z}{x+z}) H_3(\frac{z}{y}) .$$

$$(17) \quad H_4(\frac{z}{x}) = \log_e(\frac{z}{x}) = \log_e(\frac{z}{y} \cdot \frac{y}{x})$$

$$= \log_e(\frac{y}{x}) + \log_e(\frac{z}{y})$$

$$= H_4(\frac{y}{x}) + H_4(\frac{z}{y}).$$

Thus, only the logarithmic change is decomposed exactly into a sum of the component relative changes. Approximately, however, the same is also true of H_1 and H_3 when all the relative changes are small.

Let us apply equations (15)-(17) to time series (v_t), where v_t is the value of a commodity at the point of time t and equals the price p_t times the quantity q_t

$$(18) \quad v_t = p_t q_t \ .$$

We wish to express the relative change in value, $H(\frac{v_t}{v_{t-1}})$, in terms of the relative changes in price and quantity, $H(\frac{p_t}{p_{t-1}})$ and $H(\frac{q_t}{q_{t-1}})$.

For the application of (15) - (17) it will be considered that the changes in price and quantity occured step-wise:

$$p_{t-1}q_{t-1} \rightarrow p_t q_{t-1} \rightarrow p_t q_t$$

(Compare $x \rightarrow y \rightarrow z$)

$$(19) \qquad H_1\left(\frac{v_t}{v_{t-1}}\right) = H_1\left(\frac{p_t q_{t-1}}{p_{t-1}q_{t-1}}\right) + \frac{p_t q_{t-1}}{p_{t-1}q_{t-1}} H_1\left(\frac{p_t q_t}{p_t q_{t-1}}\right)$$

$$= H_1\left(\frac{p_t}{p_{t-1}}\right) + \left(\frac{p_t}{p_{t-1}}\right) H_1\left(\frac{q_t}{q_{t-1}}\right).$$

This can be written in a more transparent form by substituting $1 + H_1(p_t/p_{t-1})$ for (p_t/p_{t-1}) to get

$$(20) \qquad H_1\left(\frac{v_t}{v_{t-1}}\right) = H_1\left(\frac{p_t}{p_{t-1}}\right) + H_1\left(\frac{q_t}{q_{t-1}}\right) + H_1\left(\frac{p_t}{p_{t-1}}\right) H_1\left(\frac{q_t}{q_{t-1}}\right).$$

The last term, or the product of the relative changes in prices and quantities, will subsequently be reffered to as cross term. When the relative changes are small, this term can be neglected, and the relative change in value, $H_1(v_t/v_{t-1})$, will then equal the sum of the corresponding changes in price and quantity.

When the changes are large, the cross term will be the main term. The corresponding decomposition of H_3 is obtained as follows:

$$(21) \qquad H_3\left(\frac{v_t}{v_{t-1}}\right) = \left(\frac{p_{t-1}q_{t-1}+p_t q_{t-1}}{p_{t-1}q_{t-1}+p_t q_t}\right) H_3\left(\frac{p_t q_{t-1}}{p_{t-1}q_{t-1}}\right) +$$

$$\left(\frac{p_t q_{t-1}+p_t q_t}{p_{t-1}q_{t-1}+p_t q_t}\right) H_3\left(\frac{p_t q_t}{p_t q_{t-1}}\right)$$

$$= (\qquad) H_3\left(\frac{p_t}{p_{t-1}}\right) + (\qquad) H_3\left(\frac{q_t}{q_{t-1}}\right)$$

$$= H_3\left(\frac{p_t}{p_{t-1}}\right) + H_3\left(\frac{q_t}{q_{t-1}}\right) + \epsilon \ .$$

In this case the expression for the "cross term" ϵ is more complicated. In any event, the cross term will be one of the third degree in the relative changes. In the case of (16) the following identity, derived in Appendix 1, is obtained:

$$(22) \qquad H_3(\tfrac{z}{x}) = H_3(\tfrac{y}{x}) + H_3(\tfrac{z}{y}) - \tfrac{1}{4}H_3(\tfrac{y}{x})H_3(\tfrac{z}{y})H_3(\tfrac{z}{x}) .$$

<u>Example</u>

	x → y	y → z	x → z	"Cross term"
	10 → 12	12 → 15	10 → 15	
H_1	20.00 %	25.00 %	50.00 %	+ 5.0 %
H_3	18.18 %	22.22 %	40.00 %	- 0.4 %
H_4	18.23 %	22.32 %	40.55 %	0.0 %

The decomposition of H_3 is observed to be notably more accurate than that of H_1. This can be concluded directly from the expression for the cross term.

The changes have all been expressed as percentages relative to the mean occuring in the divisor of the indicator concerned.

Thus, in $H_1(\frac{y}{x})$ the absolute change $(y-x) = 2$ is 20 % of $x = 10$; in $H_3(\frac{y}{x})$ it is 18.8 % of $\frac{1}{2}(x+y) = 11$; and in $H_4(\frac{y}{x})$ it is 18.23 % of $L(x,y) = 10.97$.

As appeared also from the example, in the log-change the cross term is identically zero. Let us also write (17) in terms of the prices p_t and quantities q_t:

$$(23)\qquad H_4\left(\frac{v_t}{v_{t-1}}\right) = H_4\left(\frac{p_t q_{t-1}}{p_{t-1}q_{t-1}}\right) + H_4\left(\frac{p_t q_t}{p_t q_{t-1}}\right)$$

$$= H_4\left(\frac{p_t}{p_{t-1}}\right) + H_4\left(\frac{q_t}{q_{t-1}}\right)$$

For log-changes, the relative change in value invariably equals the sum of the relative changes in price and quantity.

One indicator of relative changes not yet mentioned,

$$(24)\qquad H_6\left(\frac{y}{x}\right) = \frac{y-x}{\sqrt{yx}}$$

also merits attention. H_6 is both normed and symmetric and, in addition, it has an interesting decomposition corresponding to the decomposition of H_3 in (22):

$$(25)\qquad H_6\left(\frac{z}{x}\right) = H_6\left(\frac{y}{x}\right) + H_6\left(\frac{z}{y}\right) + \epsilon, \text{ where}$$

$$\epsilon = \frac{1}{2}H_6\left(\frac{y}{x}\right)H_6\left(\frac{z}{y}\right)H_6\left(\frac{z}{x}\right)\left[\frac{1}{1+\frac{x+y}{2\sqrt{xy}}+\frac{y+z}{2\sqrt{yz}}+\frac{x+z}{2\sqrt{xz}}}\right]$$

$$\lessapprox \frac{1}{8}H_6\left(\frac{y}{x}\right)H_6\left(\frac{z}{y}\right)H_6\left(\frac{z}{x}\right).$$

The cross term ϵ of this indicator again contains the product of three relative changes, just as it should, because H_6 is symmetric. In addition, the product contains as a further factor an interesting term, involving ratios of the arithmetic and geometric means, which can well be approximated by 1/4. The derivation of this identity is presented in Appendix 1.

To summarize, for the logarithmic change H_4 the cross term is identically zero, for H_6 it is very small with the small values of the relative change, for H_3 the cross term is, to a very high degree of accuracy, numerically twice the cross term of H_6 but of the opposite sign, whereas in H_1 the cross term is of the second order of smallness, instead of the third as in the two preceding cases.

Example. The indicators will be examined numerically assuming that the relative changes are smaller than in the preceding example.

	x → y	y → z	x → z	"Cross term"
	10 → 11	11 → 13	10 → 13	
H_1	10.000 %	18.182 %	30.000 %	+ 1.818 %
H_3	9.524 %	16.667 %	26.087 %	− 0.104 %
H_4	9.531 %	16.705 %	26.236 %	0
H_6	9.535 %	16.725 %	26.312 %	+ 0.052 %

	x → y	y → z	x → z
minuend	10.000	11.000	10.000
arithmetic mean	10.500	12.000	11.500
logarithmic mean	10.492	11.973	11.435
geometric mean	10.488	11.958	11.402

Note that $L(x,y) \approx \dfrac{2\sqrt{xy}+(\frac{x+y}{2})}{3} \approx \sqrt[3]{xy(\frac{x+y}{2})} \triangleq T(x,y)$

and $H_4(\frac{y}{x}) \approx \dfrac{2H_3(\frac{y}{x})+H_6(\frac{y}{x})}{3} \approx \sqrt[3]{H_3(\frac{y}{x})^2 H_6(\frac{y}{x})}$.

This approximation to the log-change was already known by Törnqvist (1935), but Theil (1973), in particular, has used the mean T(x,y) occuring in this approximation in the index number formula he suggested. It would seem, however, that Theil has not considered the concept of a logarithmic mean to be important, since otherwise he would no doubt have derived the Vartia Indices I and II to be presented in Chapter 6. The properties of various means are considered in Appendix 3.

It is well known that the only continuous function $H:\mathbb{R}_+ \to \mathbb{R}$ satisfying the functional equation $H(xy) = H(x)+H(y)$ for all positive x and y is $H(x) = c \log_e x$ for some real constant c ; the case of differentiable H is proved in appendix 2.

We conclude that the logarithmic change

(26) $$H_4(y/x) = \log_e(y/x)$$

is the only indicator of relative change which is normed, symmetric and for which the relative change $x \to z$ is decomposed into the sum of the component relative changes $x \to y$ and $y \to z$.

The representation of the logarithmic change given in (1)

(27) $$H_4(y/x) = \log_e(y/x) = \frac{y-x}{L(x,y)}$$

will later be put in effective use in considering the determination of the relative change in a sum with the aid of the relative changes in the terms of the sum. These questions are dealt with most naturally in connection with the index numbers.

2. VARIOUS VIEWS ABOUT THE DEFINITION OF PRICE AND VOLUME INDICES

2.1. General

Like most developed human activities scientific research is pursued to satisfy various needs and purposes. A practically oriented scientist may be interested in describing, explaining, forecasting or controlling a real process under study. On the other hand, a theoretically oriented researcher (e.g., a logician, mathematician, economist or philosopher) need not have any clearly practical aims connected with empirical data or problems; he may be interested only in the theories or models used in some area. These theories often begin to live lives of their own in the researcher's mind, whose intentions may be literally philosophical. These different attitudes have contributed to the separation of 'applied' and 'theoretical' research from each other.

Analogically concepts are often classified into descriptive (or empirical) and analytical (or theoretical) ones. For instance, the arithmetic mean and the median, as calculated from observations, are descriptive measures, while the expectation and the population median are the corresponding theoretical concepts defined in the model or play process under consideration. Descriptive measures or statistics are often used as estimates of some theoretical concepts corresponding to some functions of the population parameters. Descriptive measures have, however, some meaning independently of theoretical models: they need not be interpreted as estimators of any population parameters.

A wide variety of approaches, which are not always easy to distinguish from one another, have been proposed for the construction of price and volume indices, see the classical survey of Frisch (1936) and a modern survey of Samuelson and Swamy (1974).

Nowadays most index theorists seem to prefer the analytical or theoretical approach. Many of them do not hesitate to start from the static demand or production theory, often with nonsaturation and homotheticity assumptions, and 'rationalize' index number formulas by deriving them from these and other special assumptions. These derivations show that <u>if</u> our data were generated according to their particular demand theory (where a typical consumer maximizes his time invariant utility function under given prices and income) <u>then</u> a certain index number formula would give the same numerical results as the 'true cost of living index'. This means that the index number formula may be practically useful, too.

These calculations are of course valid as such but rather irrelevant when our data is not generated in the supposed way. If p implies q but p is not true we do not know whether q is true or not. The problem is therefore whether the data is generated in the postulated way, because only then can we use the result.

It seems to be generally accepted that the standard assumptions of the static demand theory are unrealistic, so that the theory serves mostly pedagogical purposes.

Actually we are not concerned with a typical consumer but with a whole population of different economic agents. This causes the problems of aggregation over individuals, which have thus far remained unsolved [1]. If an economic agent is supposed to have his own indifference surfaces in the commodity space these would probably change in time and depend, e.g., on prices. This ruins most of the classical results in the demand theory and leads to dynamic demand functions and problems caused by taste changes [2]. Little is known of these things. The qualities of the commodities change, new ones appear and old ones disappear [3]. These complications may be theoretically handled by increasing the dimensions of the commodity space and allowing corner solutions. The stochastic nature of consumer behaviour should be taken into account by including random terms in the model [4].

An evident but essential complication stems from the fact that the adopted theory (without any other complications) would usually contain unknown parameters, so that the true cost of living index cannot be calculated unless some of these parameters are somehow estimated or fixed. Should The Statistical Officies therefore estimate their demand functions to calculate consumer price indices? What kind of demand systems should be fitted to the data? What would be the appropriate estimation procedure?

These questions are nowadays avoided by using the simplest type of descriptive price indices, namely, Laspeyres' indices, which unfortunately are usually upwards biased compared to the relevant indices in the case of demand theory. The practical man does not seem to be too serious!

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1) See e.g. Sen (1973, p. 1-23).
2) See e.g. Fisher and Shell (1972), Morishima and others (1973, p. 242-270).
3) See e.g. Törnqvist (1974), v. Hofsten (1952), Griliches (Ed.) (1971).
4) See e.g. Theil (1965, 1967, p. 227-289, 1970), Deaton (1974), Diewert (1974).

As at least some of these complications are widely admitted to be relevant it seems quite remarkable that so much interest is still focused on the standard mathematical game of the index theorists, which ignores these complications. While Samuelson and Swamy (1974) declare their belief in the economic theory in their Concluding Warning, I want instead to study the index number problem from the descriptive point of view.[1)]

2.2. The descriptive approach

The most generally applicable approach is the descriptive approach, for which Frisch (1936) uses the term 'atomistic approach' and which Samuelson calls 'statistical'. Here a price index is constructed to measure the change in the 'price level' (or the average change in individual prices), very little being asserted about the behaviour of individual prices and quantities. This is mainly the approach of the pioneers in the field, e.g., Walsh, Jevons, Laspeyres, Paasche, Sauerbeck, Edgeworth and, especially, Irving Fisher.

In the descriptive approach we distinguish a historically interesting old way of thinking, namely, a stochastic approach. The earliest writers tended to conceive of price indices as some kinds of means or measures of central tendency of the universe of price changes. Individual price changes were regarded as random observations of this hypothetical universe. The purpose of the index formula was to eliminate the random fluctuations of individual price quotations. In the stochastic approach observed individual price or volume changes were considered to

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1) Dr. Pentti Vartia has called the descriptive approach 'theory invariant' in contradistinction to to the 'theory dependent' economic approach. This description nicely stresses the robustness of the descriptive approach, which does not use so many (doubtful) assumptions as the economic theory. Of course both approaches have some common theoretical elements, e.g., values, quantities and prices are supposed to be measured on ratio scales.

give unreliable information on the average price change, roughly in the way measurement errors hide the true effects in experimental situations.

Sometimes one may meet the similar opinion that individual price ratios are something unreliable and that only their averages, i.e., price indices, give some reliable information. We must state clearly, however, that this approach is only of historical interest and that the definition of a price index cannot logically be based on such foundations. For instance Keynes (1930) p. 85 vigorously criticizes the stochastic definition of a price index as being 'root-and-branch erroneous'. Frisch (1936) agrees with Keynes on this point.

I. Fisher's "The Making of Index Numbers" is a landmark in the descriptive approach. Before Fisher's work there had been a long controversy over the proper index number formulas. The chief argument, which I. Fisher wanted to reject, was that one index number was fit for one purpose and another for another one. We cite Fisher (1922) p. 231: "Unless someone has the hardihood to espouse bias or freakishness for some "purpose" whatever formula he advocates will insist on coinciding with whatever formula anyone else advocates." For a reconsideration of the concept of 'bias' as used by Fisher see Y. Vartia (1976b).
Fisher's method is based on definite criteria or 'tests' which a good index number formula should satisfy. The most fundamentally important test among those already treated in his earlier study, Fisher (1911), is the 'time reversal' test. "This and the new test, the 'factor reversal' test, are here constituted the two legs on which index numbers can be made to walk", Fisher (1922) p. XIII.

Following Frisch (1936), we may characterize this approach to the construction of index numbers as test theoretic.

In this approach an index number formula is a function meeting certain desirable criteria or tests. The test theoretic approach is transformed into an axiomatic approach by only formalizing the tests and calling them axioms. As was proved[1] by Swamy (1965) there are no index number formulas satisfying all of Fisher's tests; i.e., the corresponding axioms are contradictory and the set of indices satisfying them is empty. This situation resembles the problem now known as 'Arrow's paradox'. It is concerned with the construction of society's preference function out of the preference functions of individuals. By setting up a certain seemingly natural list of properties (axioms) which this construction should satisfy, Arrow (1963) proved that no such construction was possible.

As was humorously pointed out to me by L. Törnqvist, there should be nothing especially astonishing in the fact that a set can be made empty by increasing the list of properties its elements should possess.

In the index number problem (and in the problem discussed under the heading of Arrow's paradox)[2] there have been at least two ways of overcoming the seeming paradox caused by the empty sets.

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1) E.g. Frisch (1930) and Wald (1937) have presented their "proofs", which according to Swamy (1965) suffer from some errors.

2) See Luce & Raiffa (1966) pp. 340-1, 356-7.

The first is to drop some of the desired properties of the index number formulas (i.e., functions $I: A \to B$), while their definition set A is kept unchanged. Fisher dropped, for instance, the 'circular property' from his list of desirable properties. We cite Fisher (1922) p. 271: "But the analogy of circular test with time reversal test, while plausible, is misleading. I aim to show that the circular test is theoretically a mistaken one, that a necessary irreducible minimum of divergence from such fulfilment is entirely right and proper, and, therefore, that a perfect fulfilment of this so-called circular test should really be taken as proof that the formula which fulfils it is erroneous".

Fisher's conclusion is not accepted by some modern researchers, who usually base their views on a different starting point, namely the economic theory of index numbers. Samuelson and Swamy (1974) p. 575 wrote, in commenting this conclusion of Fisher's: "Alas, Homer has nodded; or, more accurately, a great scholar has been detoured on a trip whose purpose was obscure from the beginning".

It is not known what are the desirable properties making Fisher's "Ideal Index" (or its rivals) the only index having these properties ; i.e., we do not have different axiom systems (in this descriptive approach) that would characterize one and only one index formula each. Fisher's work was so impressive that only little has been added to his results on the test theoretic approach.

2.3. The economic approach

The second way of avoiding the no-solution case could be called the economic theory of index numbers or the economic approach.

Instead of dropping some desirable properties of the index number formulas, we restrict the admissible combinations of prices and quantities by some economic theory (e.g., the static demand theory or the production theory). By 'imagining' certain interdependencies between prices and quantities we restrict the definition set A of the "index formula" I: A→B to a much smaller but more complicated set A' ⊂ A. The index number formula - e.g., the 'true cost of living index' - will then be defined using the concepts of the underlying economic theory. In special circumstances the index number formula can be proved to have desirable properties in the restricted set A' (but not, of course, in A), see Samuelson and Swamy (1974).

In the economic approach the price index has a definite economic meaning: it answers a definite question. But this will be the case only if the underlying economic theory is a true description of the data generating process. In a more general situation discussed in the descriptive approach, the definitions of the economic theory have no clear meaning.

We are in a very problematic situation: in order to escape the inconsistencies of the descriptive approach ("What is the right index formula?") we hypothesize a complicated economic theory, which can be regarded only as an approximation to the real data generating process. If the approximation is a poor one, our index number calculations may be totally misleading unless they can be interpreted using the descriptive approach, which is luckily often the case. When the definitions of the economic theory are carelessly used in real situations, more complications will perhaps be introduced than eliminated:

the best may be an enemy of the good. The usefulness of the economic approach (and all idealized play processes or theories) lies in my opinion in the fact that it clarifies, like good examples, our conception of the more complicated situations met in the real world. By examining a beautiful but restricted theoretical world we may learn what are the minimum complications found in a wider world.

To show some problems in the definition of price and volume indices in the economic theory, we use the following definitions given by Samuelson and Swamy (1974):

DEFINITION: Economic Price Index: This must equal the ratio of the (minimum) costs of a given level of living in two price situations.

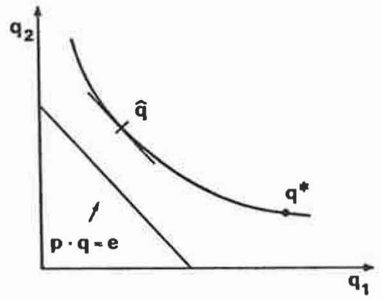
The definition is concerned with only "a given level of living", which means that the Economic Price Index is not exclusively a function of two price situations but usually depends on the given level of living. This definition gives two natural but usually different price indices when we want to compare prices in two situations involving different levels of living. We quote Samuelson and Swamy (1974) p. 568:

> "The fundamental and well known theorem for the existence of a price index that is invariant under change in level of living Q^a, is that each dollar of income be spent in the same way by rich or poor, with all income elasticities exactly unity (the homothetic case). Otherwise, a price change in luxuries could affect only the price index of the rich while leaving that of the poor relatively unchanged. This basic theorem was well known already in the 1930's, but is often forgotten and is repeatedly being rediscovered".

The indices based on economic theory can be defined in different notations by using, e.g., utility functions, demand functions or indirect utility functions. In the following we will use the indifferenc relation instead of the utility function, adopting elsewhere the concepts of Samuelson and Swamy (1974).

Let q^* be a given bundle of commodities and q any bundle of commodities indifferent to it for our consumer, $q \sim q^*$. Let $e(p; q^*)$ be the minimum expenditure needed to buy a consumption vector q indifferent to q^* when prices are p:

$$e(p; q^*) = \min_{q \sim q^*} p \cdot q = p \cdot \hat{q}$$

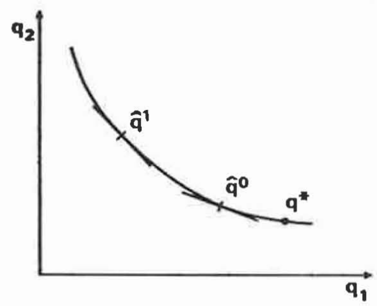


By its definition, the Economic Price Index can be written as

$$P(p^1, p^0; q^*) =$$

$$e(p^1; q^*)/e(p^0; q^*) =$$

$$p^1 \cdot \hat{q}^1 / p^0 \cdot \hat{q}^0$$



As Samuelson and Swamy (1974) prove $P(p^1, p^0; q^*)$ is independent of the utility level fixed by q^* only if the indifference contours $S(q^*) = \{q | q \sim q^*\}$ are homothetic with all income elasticities exactly unity. This guarantees that value shares do not depend on income. The Economic Price Index satisfies the strong proportionality test $P(kp^1, p^0; q^*) \equiv kP(p^1, p^0; q^*)$ for any positive k.

The definition of the quantity index is more complicated.[1)]

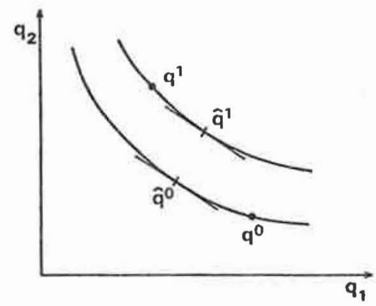
DEFINITION: Economic Quantity Index: This measures for two presented quantity situations q^0 and q^1, the ratio of the minimum expenditure needed, in the face of a reference price situation p^*, to buy their respective levels of well-being.

Or in mathematical notation

$$Q(q^1,q^0;\ p^*) =$$

$$e(p^*;\ q^1)/e(p^*;\ q^0) =$$

$$p^*\cdot\hat{q}^1/p^*\cdot\hat{q}^0$$



Just like the Economic Price Index, the Economic Quantity Index will only in the homothetic case be the same for any chosen price standard p^*. A curiosity is that the proportionality test, $Q(kq^0,q^0;\ p^*)\equiv k$ for any positive k, is satisfied only in the homothetic case, see Samuelson and Swamy (1974) p. 576 or Y. Vartia (1976d).

Let (p^0,q^0) and (p^1,q^1) be two equilibrium situations. Specifying $q^* = q^0$ and $p^* = p^1$ we get

(1) $$P(p^1,p^0;\ q^0)\ Q(q^1,q^0;\ p^1) = p^1\cdot q^1/p^0\cdot q^0 .$$

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1) There are different definitions of the quantity index mentioned, e.g., by Samuelson and Swamy (1974) p. 590 note 17. and p. 591. Pollak's definition happens to be the same as Leontief's (1936) definition to be formalized later.
For the definition of a marginal price index, see Rajaoja (1958) or Theil (1967). Rajaoja also defines a very interesting index, the price index of the competitors of the good G_k, and proves useful theorems. These concepts can be defined in a natural way only using economic theory and we do not discuss them here.

This corresponds to a similar identity between Laspeyres' price and Paasche's quantity indices. The identity tells us that, using the price index corresponding to the utility level of q^0 and deflating the value ratio by it, we would get not $Q(q^1,q^0;\ p^0)$ but $Q(q^1,q^0;\ p^1)$, where the new price vector p^1 changes from one situation to another.
By specifying $q^* = q^1$ and $p^* = p^0$ we get an identity which corresponds to Paasche's price and Laspeyres' quantity indices:

(2) $$P(p^1,p^0;\ q^1)\ Q(q^1,q^0;\ p^0) = p^1\cdot q^1/p^0\cdot q^0 \quad .$$

This equation shows an asymmetry similar to that in (1).

Next we derive two alternative one-sided bounds for Economic Price Indices of Laspeyres' type $P(p^1,p^0;\ q^0) = e(p^1;\ q^0)/p^0\cdot q^0$ and Paasche's type $P(p^1,p^0;\ q^1) = p^1\cdot q^1/e(p^0;\ q^1)$.
Because $e(p^1;\ q^0)$ is the minimum expenditure necessary to attain the indifference surface determined by q^0 we have $e(p^1;\ q^0) \leq p^1\cdot q^0$. Here $p^1\cdot q^0$ is the total income needed to buy q^0 basket at p^1 prices. Therefore we have for Laspeyres' price index L_p:

(3) $$P(p^1,p^0;\ q^0) \leq p^1\cdot q^0/p^0\cdot q^0 = L_p \quad .$$

Similarly we get for Paasche's price index P_p:

(4) $$P(p^1,p^0;\ q^1) \geq p^1\cdot q^1/p^0\cdot q^1 = P_p \quad .$$

These one sided bounds can be combined into a double limit generally only in the homothetic case where the left hand sides of (3) and (4) are equal. In the nonhomothetic case it is even possible to have

(5) $P(p^1,p^0;\ q^0) \leq L_p < P_p \leq P(p^1,p^0;\ q^1)$,

so that the Economic Price Index need not lie between L_p and P_p.

Similar bounds are easily derived for Economic Quantity Indices as well:

(6) $Q(q^1,q^0;\ p^0) \leq p^0 \cdot q^1 / p^0 \cdot q^0 = L_q$

(7) $Q(q^1,q^0;\ p^1) \geq p^1 \cdot q^1 / p^1 \cdot q^0 = P_q$.

These bounds show that Laspeyres' price and quantity indices L_p and L_q are usually upwards biased with respect to Laspeyres' type of Economic Indices, while Paasche's indices P_p and P_q are analogously downwards biased.[1)] These or other but similar bounds (see Frisch (1936) p. 24) have been discussed in numerous articles, and confusion is here no rarity as mentioned, e.g., by Frisch (1936) p. 25-26 and Samuelson and Swamy (1974) p. 581. The revealed preference theory is based on these inequalities, see e.g. Samuelson (1947), Houthakker (1950) and Afriat (1967).

Note that the bounds (6) and (7) need not hold for all choices of reference prices p^* but only for the special choices p^0 and p^1. The same applies to the price index $P(p^1,p^0;\ q^*)$. This seems to be the point stressed by Leontief (1936), which we shall comment on later.

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1) In the case of production theory the inequalities are reversed, see Samuelson and Swamy (1974) p. 589 and Fisher and Shell (1972) p. 58.

Laspeyres' and Paasche's types of price indices $P(p^1,p^0; q^0)$ and $P(p^1,p^0; q^1)$ differ usually somewhat from each other because they have been calculated on different utility levels. Their geometric mean is a Fisher type of price index, which may be interpreted as a price index corresponding to an utility level specified by an intermediate consumption vector $\bar{q}$:

(8) $$\sqrt{P(p^1,p^0; q^0)\ P(p^1,p^0; q^1)} = P(p^1,p^0; \bar{q}),$$

where $\bar{q}_i \approx \sqrt{q_i^0\ q_i^1}$. In the same way the quantity index should be calculated using an intermediate price vector $\bar{p}$, $\bar{p}_i \approx \sqrt{p_i^0\ p_i^1}$, such that

(9) $$P(p^1,p^0; \bar{q})\ Q(q^1,q^0; \bar{p}) = p^1 \cdot q^1 / p^0 \cdot q^0 \quad .$$

Vectors $\bar{q}$ and $\bar{p}$ exist by the mean value theorem: A continuous function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ assumes all values between $f(x^1)$ and $f(x^2)$ on any continuous arc connecting x^1 and x^2.

These indices are symmetric in respect to situations 0 and 1, but in time series analysis the reference quantities $\bar{q}$ and prices $\bar{p}$ change in time. Therefore, e.g., in time series studies $P(p^1,p^0; \bar{q})$ for different p^1:s is not literally a 'constant utility price index'. These problems are often ignored in literature

Our analysis is in the spirit of Theil (1967). Theil proved that the Törnqvist type of price index approximates quadratically the Economic Price Index calculated on the utility level corresponding to the geometric mean of real incomes. As the best index number formulas - e.g. Fisher's Ideal Index, Törnqvist's index, our new indices etc. - may be shown to approximate each other quadratically for small relative changes in prices and quantities, they all give quadratical approximations to the Economic Price Index calculated at the geometric mean of real incomes, see Diewert (1976b) and Y. Vartia (1976b) for newer results.

I interpret these results as speaking for the descriptive approach. They show, it seems to me, that index formulas based on a completety descriptive approach are able to contend successfully with the Economic Indices on the home field of the latter: in the ideal world of the demand theory.

It is impossible to arrange a similar contest on the home field of descriptive indices (in the general situation, where the prices and volumes change freely), as the Economic Indices simply cannot be transferred to this more general world. Their definition presupposes a connection between prices and quantities given by economic theory.

2.4. On Leontief's quantity index

To illustrate some problems and different interpretations I want to analyse a critical article by Leontief (1936), which seems to contain profound but often neglected results. By formalizing Leontief's notion of a quantity index we get a different definition for this general concept. This should show that even in the economic theory of index numbers there is no complete agreement about the 'best definitions'.

Leontief's conclusions are radical: Economic price and quantity indices are nonmeasurable magnitudes, which can be described only as being larger or smaller than one. We cite Leontief (1936) p. 48:

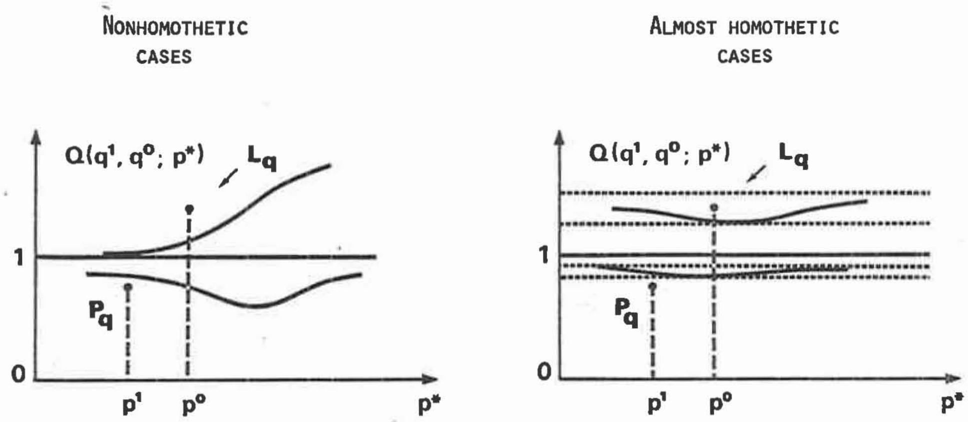
> "A method of index number calculation based solely on a given 'system of tastes' as represented by a succession of indifference lines cannot possibly lead to any other result than a series of non-measurable magnitudes. If, notwithstanding, these results are given in the form of definite numbers, we have to discard their numerical meaning entirely and take into account only the respective order of magnitude. In so far as such an index number represents a ratio between two composite prices or quantities, its economic significance, if it exists at all, can be represented in terms of one of the three signs: >1, <1 or = 1 (or using percentages: >100 per cent, <100 per cent, = 100 per cent). Any further numerical definitess which an index number seems to convey is devoid of economic meaning. No wonder that every attempt toward a numerical interpretation, in the given circumstances, produce nothing but confusion".

According to Leontief's view a statement like "the volume of consumption is 10 % greater than last year" means only that consumption has increased. Leontief's criticism is based on the ordinal character of the utility function. He does not present his conclusions as mathematical theorems but characterizes them mostly verbally. These characterizations are perhaps excessively radical and easily misunderstood.

Much of Leontief's analysis is based on lower and upper limits of $Q(q^1,q^0; p^*)$. His table 1 intends to express how little can be inferred about the _entire function_ $Q(q^1,q^0; p^*)$ _of the variable_ p^* for a fixed pair (q^1,q^0) when (p^0,q^0) and (p^1,q^1) are equilibrium points and we know only which of the nine possible cases $L_q = p^0 \cdot q^1/p^0 \cdot q^0 \gtreqless 1$ and $P_q = p^1 \cdot q^1/p^1 \cdot q^0 \gtreqless 1$ applies. For instance if $L_q = 1.0095 > 1$ and $P_q = 0.9745 < 1$ then we know that

$\Omega(q^1,q^0;\ p^0) \leq 1.0095$ and $\Omega(q^1,q^0;\ p^1) \geq 0.9745$. Thus the function $\Omega(q^1,q^0;\ p^*)$ of p^* may be for all p^* smaller than 1, for all p^* greater than one or identically one. One of the cases applies to a fixed system of tastes.

SCHEMATIC REPRESENTATION FOR EQUATIONS (6) AND (7).



Leontief gives his conclusions in his table 1 (notation ours):

<u>Leontief's table 1</u>

	"Upper limit" $L_q=p^0\cdot q^1/p^0\cdot q^0$	"Lower limit" $P_q=p^1\cdot q^1/p^1\cdot q^0$	The magnitudes of the "true" quantity index
I	1	1	1
II	1	>1	>1
III	1	<1	<1
IV	>1	1	>1
V	>1	>1	>1
VI	>1	<1	$\gtrless 1$ (Indeterminate)
VII	<1	1	<1
VIII	<1	>1	$\gtrless 1$ (Indeterminate)
IX	<1	<1	<1

However, Leontief's cases II, VII and VIII are impossible if the data is generated according to the demand theory.

This follows from equations (6) - (7) and the fact that $Q(q^1,q^0; p^*)$ is for a fixed pair (q^1,q^0) and for all p^*:s on the same side of 1 unless it is identically one. Note that this does not apply to $P(p^1,p^0; q^*)$, which may well cross the 1-surface.[1)] Cases II and VIII are wrongly classified also in the Revealed Preference table given by Diewert (1976a):

Revealed Preference Table I:

	$P_q \leq 1$ or $p^1 \cdot q^1 \leq p^1 \cdot q^0$	$P_q > 1$ or $p^1 \cdot q^1 > p^1 \cdot q^0$
$L_q < 1$ or $p^0 \cdot q^1 < p^0 \cdot q^0$	q^0 revealed preferred to q^1: $q^1 \prec q^0$	Inconsistent preferences have been revealed
$L_q \geq 1$ $p^0 \cdot q^1 \geq p^0 \cdot q^0$	Zone of Indeterminacy	q^1 revealed preferred to q^0: $q^1 \succ q^0$

1) This is an example of the nonsymmetry ("nonduality") between prices and quantities in the demand theory.

A complete crosstabulation of the nine cases gives us a sharper picture of the situation:

Revealed Preference Table II:

	$P_q < 1$	$P_q = 1$	$P_q > 1$
$L_q < 1$	$q^1 < q^0$	Incon-sistent prefe-rences	Inconsistent preferences
$L_q = 1$	$q^1 \lesssim q^0$	$q^1 \sim q^0$	Inconsistent preferences
$L_q > 1$	Zone of Indeterminacy	$q^1 \gtrsim q^0$	$q^1 > q^0$

In Leontief's case I we have $q^1 \sim q^0$, which sharpens the Zone of Indeterminacy in Table I. In Leontief's case IV, $L_q > 1$ and $P_q = 1$, (which is possible in the case of demand theory) we know that $Q(q^1,q^0;\ p^0) \leq 1+\epsilon$, where $\epsilon = L_q - 1 > 0$, and $Q(q^1,q^0;\ p^1) \geq 1$. Therefore $\forall p^*$: $Q(q^1,q^0;\ p^*) \gtrsim 1$ or $q^1 \gtrsim q^0$. It is possible that $q^1 \sim q^0$ here, i.e., Leontief's proposition is too strong.

In the same way $L_q = 1$ and $P_q < 1$ implies $q^1 \lesssim q^0$, also here Leontief claims too much. Note that we succeeded in evaluating more exactly the boarderline of the Zone of Indeterminacy[1)] in Table I.

1) It seems that in Leontief's example on p. 51 the numbers or the symbols or both are badly mixed up.

Frisch (1936) states on p. 21 correctly (if inconsistent preferences are excluded) that $P_q > 1$ implies $q^1 \succ q^0$ and $L_q < 1$ implies $q^1 \prec q^0$. Thus far everything is comprehensible, but Leontief claims more (p. 50):

> "The "true" quantity index, even if successfully obtained, in general has still no definite numerical meaning. It is a magnitude defined solely in its relation to unity."

I agree, if the meaning of the proposition is interpreted as follows: Let $f(p^*)$ be a function which is either (A) always > 1 or (B) always < 1 or (C) always = 1. Then we can infer

(A) $f(p) > 1$ for some $p \Rightarrow$
$f(p^*) > 1$ for all p^*

(B) $f(p) < 1$ for some $p \Rightarrow$
$f(p^*) < 1$ for all p^*

(C) $f(p) = 1$ for some $p \Rightarrow$
$f(p) = 1$ for all p.

It might happen for some such f and p^0 that $f(p^0) = 1.25$ but for every $\epsilon > 0$ there exists a p such that $1 < f(p) < 1 + \epsilon$.

But I must disagree, if Leontief's proposition is interpreted to mean, e.g., that in <u>any</u> situation (say in my fixed indifference system describing the private consumption of 30 consumption categories in Finland 1976-77) a result such that $Q(q^1, q^0; p^0) = 1.25$ means no more than say $Q(q^1, q^0; p^0) = 1.01$; i.e., only that $q^1 \succ q^0$.

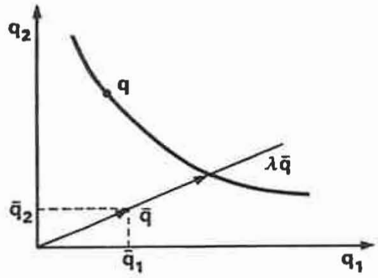
The disagreement arises because my indifference system may be such that from $Q(q^1,q^0; p^0) = 1.25$ I can infer that $\forall p^*$: $1.2 \leq Q(q^1,q^0; p^*) \leq 1.3$ and not only that $\forall p^*$: $Q(q^1,q^0; p^*) > 1$. In contrast to Leontief I would like to say: The true quantity index $Q(q^1,q^0; p^*)$ has for any p^* a completely definite numerical meaning and the totality of its values for a given pair (q^1,q^0) gives us the numerical limits between which any quantity comparison lies.

It seems evident that Leontief's verbal explanation is too strong and apt to be missunderstood. Had Leontief meant something about which I just expressed my disagreement above, he would probably not have presented his geometric construction of the quantity index, which we next want to represent algebraically. Leontief considers three commodities A, B and C and shows how, by defining a new composite commodity I containing a fixed proportion of B and C, the number of commodities is reduced to two, namely A and I. Two commodities may be represented similarly by a single composite commodity.

The idea in his construction is as follows. Choose first an arbitrary relative combination of the commodities represented by the vector $\bar{q} = (\bar{q}_1,...,\bar{q}_n)$, where all $\bar{q}_i > 0$. This vector $\bar{q}$ represents a new composite commodity, which contains a fixed proportion of the original commodities.

Next determine how many 'units' λ of $\bar{q}$ is needed to make our consumer indifferent between $\lambda\bar{q}$ and a given consumption vector q:

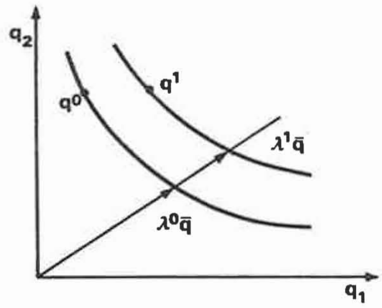
Find $\lambda>0$ such that $\lambda\bar{q}\sim q$.



This is a generalization of a method used, e.g., in the analysis of energy consumption in which the total consumption of different categories of energy is expressed, say, in coal equivalents. Here the equivalence is defined usually by the energy contents and fixed transformation coefficients are used.

We may express any two consumption vectors q^0 and q^1 in terms of our $\bar{q}$-equivalents: $\lambda^0\bar{q}\sim q^0$ and $\lambda^1\bar{q}\sim q^1$. Leontief's quantity index $Q(q^1,q^0|\bar{q})$ is then given by

$$Q(q^1,q^0|\bar{q}) = \lambda^1/\lambda^0$$



Leontief's index[1)] is the ratio λ_1/λ_0 between the amounts of the 'composite commodity' $\bar{q}$ which would just compensate the commodity bundles q^0 and q^1.

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1) After writing this I discovered from Samuelson and Swamy (1974), note 17, that this is just Pollak's definition of the quantity index, which he has given in an unpublished paper. Malmqvist (1953) has given still another but a very similar definition, see e.g. Diewert (1976b). All these definitions have their 'dual' price analogs so that the economic theory has not succeeded in giving only one 'right' definition for the 'true indices'.

$Q(q^1,q^0|\bar{q})>1$, $= 1$ or <1 according to whether $q^1 \succ q^0$, $q^1 \sim q^0$ or $q^1 \prec q^0$ for any choice of index commodity $\bar{q}$. This means that $Q(q^1,q^0|\bar{q})$ is itself a 'cardinal indicator of utility'; i.e., $Q(q,q^0|\bar{q})$ is for fixed q^0 and $\bar{q}$ an increasing function of the utility function $u(q)$ and likewise $Q(q^1,q|\bar{q})$ is for fixed q^1 and $\bar{q}$ a decreasing function of $u(q)$, compare Samuelson and Swamy (1974) p. 568.
It is easily seen that $Q(q^1,q^0|\bar{q})$ is independent of $\bar{q}$ only in the homothetic case. In this simple and unrealistic situation Leontief's point disappears: in the homothetic case we can express in exact figures the changes in the quantity of consumption. This reveals that Leontief's verbal conclusions are not always true. There exist systems of preferences for which they are valid but usually more can be said about price and volume indices than Leontief claims.
If the indifference surfaces $S(q^*) = \{q|q \sim q^*\}$ are 'almost homothetic', the quantity index $Q(q^1,q^0|\bar{q})$ must be 'almost quantitative'; i.e., for a given pair (q^1,q^0) it varies only little when $\bar{q}$ is changed. For instance it may happen that $Q(q^1,q^0|\bar{q}) \in [1.2, 1.3]$ when $\bar{q}$ gets all its admissible values and (q^1,q^0) is a given pair. Doesn't this mean not only that consumption has increased $(q^1 \succ q^0)$ but also that it has increased at least by 20 % but by not more than 30 %?

To analyse the situation we will approximate the upper[1)] indifference surface $S(q^1) = \{q|q \sim q^1\}$, where $q^1 \succ q^0$, both from below and above

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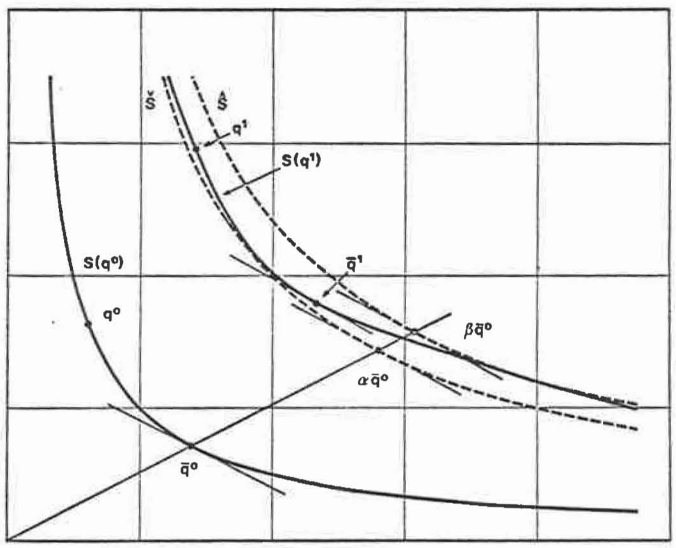
1) By analysing conversely the lower indifference surface we get just the converse results.

by surfaces $\check{S}$ and $\hat{S}$ of the same form as the lower indifference surface $S(q^0)$. Define first $\alpha = \inf Q(q^1,q^0|\bar{q})$ and $\beta = \sup Q(q^1,q^0|\bar{q})$ over all admissible $\bar{q}$:s. Leontief's quantity index then lies between these limits. As Leontief stresses, in the general case we can prove only that $1 \leq \alpha \leq \beta$, in the homothetic case, however, we have $1 < \alpha = \beta$. Usually $1 < \alpha < \beta < \infty$. Now define $\check{S} = \{q | \exists \bar{q}^0 : \bar{q}^0 \sim q^0 \ \& \ q = \alpha\bar{q}^0\}$ and $\hat{S} = \{q | \exists \bar{q}^0 : \bar{q}^0 \sim q^0 \ \& \ q = \beta\bar{q}^0\}$ so that for all $\bar{q}^0 \in S(q^0)$, $\check{q} \in \check{S}$, $\bar{q}^1 \in S(q^1)$ and $\hat{q} \in \hat{S}$ we have $\bar{q}^0 \precsim \check{q} \precsim \bar{q}^1 \precsim \hat{q}$.

If $1 < \alpha < \beta < \infty$ then $\bar{q}^0 \prec \check{q} \precsim \bar{q}^1 \precsim \hat{q}$. We will next show that the Economic Quantity Index $Q(q^1,q^0|p^*)$, too, lies between α and β for any p^*.

Take some p^* and let $Q(q^1,q^0;p^*) = e(p^*;q^1)/e(p^*;q^0) = p^* \cdot \bar{q}^1 / p^* \cdot \bar{q}^0$, where $\bar{q}^1 \sim q^1$ and $\bar{q}^0 \sim q^0$.



For $1 \leq \alpha \leq \beta$ we have $\bar{q}^0 \precsim \alpha\bar{q}^0 \precsim \bar{q}^1 \precsim \beta\bar{q}^0$ because they lie respectively on $S(q^0)$, $\check{S}$, $S(q^1)$ and $\hat{S}$. Taking inner products with p^* we get $p^* \cdot \bar{q}^0 \leq \alpha(p^* \cdot \bar{q}^0) \leq p^* \cdot \bar{q}^1 \leq \beta(p^* \cdot \bar{q}^0)$. The two last inequalities result from the fact that the budget plane $e = p^* \cdot q$ becomes tangent to $\check{S}$, $S(q^1)$ and $\hat{S}$ in this order. The proposition follows by dividing by $p^* \cdot \bar{q}^0$.

2.5. Conclusion

I have briefly reviewed the main approaches to the index number problem. The descriptive (or atomistic, statistical) approach is usually divided into stochastic and test theoretic (or essensially axiomatic) approaches. In this approach the problem is investigated with no substantial help from any subject matter theories, using only logic, mathematics, statistics and other method sciences. On the contrary the economic (or functional) approach regards the index number problem as a part of some economic theory and therefore the proposed solutions waver[1] together with the mother theory.

The subject matter of this chapter is highly controversial and if the different approaches are difficult to separate from each other, they are even more difficult to evaluate. The descriptive approach is more robust than the economic approach, but the other side of the coin is that the latter yields more exact and clearly interpretable results if the underlying hypothesis happens to be a correct one. The efficiency of a procedure is, of course, increased if more correct information is used, the problem being in the correctness of the 'information', cf Teräsvirta and Vartia (1975,p.6).

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1) If a solution does not find enough support in the descriptive approach, it falls together with the mother theory. If we choose, e.g. a special case of the Cobb-Douglas utility function as our utility function, $u(q) = (q_1 \cdot q_2 \ldots q_n)^{1/n}$, then the Economic Price Index $P(p^1,p^0; q^*)$ is the unweighted geometric mean of the price ratios (p_i^1/p_i^0). I would not draw, like Swamy (1965), any general conclusions from this index number formula, which falls together with its naive mother theory.

A given method of index calculation - e.g. Fisher's index used as a base index - may be evaluated using both approaches and this method scores high in both cases. Therefore Fisher's index does not need support from economic theory to be applicable. Other methods may seem still better from some or all angles.

Our aims is to develop further the descriptive approach, in order to see how far we can get without anchoring our results to any particular economic theory.

3. THE DEFINITION OF INDEX NUMBER FORMULAS AND CERTAIN DESIDERATA CONCERNING THEM

3.1. General

Irving Fisher, one of the pioneers of the study of price and volume index numbers, developed numerous excellent index number formulas and showed, on the other hand, the uselessness of several other generally employed index number formulas. Here, an attempt will be made to define axiomatically the concept of an index number formula in such a way that at least obviously useless formulas will remain outside its scope. Our approach is descriptive as contrasted to the economic approach and we try to represent axiomatically the main ideas of Fisher's (1922) 'test approach'. Our approach is in the spirit of Swamy (1965) and especially Eichhorn (1976), although neither of them tries to define the general concept of an index formula.

Consider the commodities a_i, $i=1,\dots,n$, which are supposed to be perfectly homogeneous and of equal quality in the course of examination. We suppose that a complete set of prices p_i, quantities q_i and values $v_i = p_i q_i$ are known for the set of commodities $A = \{a_1,\dots,a_n\}$:

commodities:	$a_1, a_2, \ldots, a_n$
prices:	$p_1, p_2, \ldots, p_n$
quantities:	$q_1, q_2, \ldots, q_n$
values:	$v_1, v_2, \ldots, v_n$

Let us imagine that a_i:s are commodities sold at a commodity exchange and that the data has been converted so as to correspond to the sales during a given period (e.g., one year). The prices are unit prices, such as 53 pennies/litre, the quantities are expressed in physical units or are dimensionless numbers and the values $v_i = p_i q_i$ are expressed in monetary units. The total sales of the exchange in these commodities is

(1) $$V = \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} p_i q_i = p \cdot q \quad ,$$

using the inner product notation for vectors p and q.
The value share of a_i is denoted by $w_i = v_i/V$.

Assume that the data in question are known for two time periods, 'yesterday' t_0 and 'today' t_1, which need not necessarily be equally long, provided that the data has only been converted so as to correspond to, say, a period of a year's length. The variables from different periods are indicated by superscripts.

The aim is to define the "price" P and "volume" Q of the total sales in such a way that, for both periods,

(2) $$V^k = \Sigma p_i^k q_i^k = P^k Q^k \quad , \quad k = 0,1 \quad .$$

and that the ratios

(3) P^1/P^0 and Q^1/Q^0 ,

which are independent of the units of measurement, will indicate the changes that have occurred in the average prices and quantities. Provided that the decomposition (2) is known, the ratios (3) can immediately be computed. On the other hand, the ratios (3) do not uniquely determine the decomposition (2), which only yields P and Q up to a multiplicative constant ("unit of measurement").

If the 'price and volume indices' in (3) are defined in a reasonable way and a decomposition $V^0 = P^0Q^0$ is determined in one way or another, we may calculate P^1 and Q^1 from

(4) $P^1 = P^0(P^1/P^0)$, $Q^1 = Q^0(Q^1/Q^0)$.

It should be stressed, however, that the determination of a ratio is not a distinctive characteristic of an index number; instead, the problem of index numbers consists in how the decomposition (2) should reasonably be defined. Therefore, I do not consider it appropriate to call the <u>price relative</u> p_i^1/p_i^0 of a commodity a_i a price index, as it is sometimes called.

The usual approach is to try to define the indices P^1/P^0 and Q^1/Q^0 directly. The idea underlying this approach is to determine the <u>price level</u>, and the so-called <u>ideal index</u>,

(5) $P^1/P^0 = \sqrt{\left(\frac{p^1 \cdot q^0}{p^0 \cdot q^0}\right) \left(\frac{p^1 \cdot q^1}{p^0 \cdot q^1}\right)}$

strongly recommended by Fisher (1922), is perhaps the index most generally regarded as the one best suitable for the purpose.

Another way of thinking, and a more fruitful one in my opinion, is to try to determine not the price level but the relative change in prices, as did Divisia (1925), Törnqvist (1936), Theil (1973). The best indicator of the relative change is the logarithmic change, with the aid of which (1) may be expressed as

$$(6) \qquad \log(V^1/V^0) = \log(P^1/P^0) + \log(Q^1/Q^0) \quad .$$

Törnqvist, for example, defined the log-change of the price index as a weighted average of the individual log-changes in prices

$$(7) \qquad \log(P^1/P^0) = \sum \frac{C_i \ \log(p_i^1/p_i^0)}{\Sigma C_i} \quad ,$$

where the weight C_i is in practice defined as an arithmetic average of either old and new values or value shares.[1)] Other specifications of (7) and their variations will be introduced and discussed later.

Next a mathematical definition of a index number formula will be given, formulated in such a way that certain inappropriate formulas, such as the "price index formula"

$$(8) \qquad \Sigma p_i^1/\Sigma p_i^0$$

sometimes employed, will be excluded. On the other hand, the formula used by The Economist in 1927-1958,

$$(9) \qquad (\prod_{i=1}^{n} (p_i^1/p_i^0))^{\frac{1}{n}}$$

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1) In Finland this formula has been used, e.g., by the following: The Bank of Finland, see Törnqvist (1937) , The Post and Telegraph Office, see Törnqvist (1971), the State Alcohol Monopoly, see Nyberg (1967) and in some special studies published by The Central Statistical Office, see Somervuori (1972)

based on price relatives does qualify for a price index formula. The price and volume indices could be denoted as formal ratios (3) but to avoid possible confusion we will denote them as P_{t0}^{t1} and Q_{t0}^{t1} (or shortly P_0^1 and Q_0^1).

3.2. The definition of index number formulas

Fisher (1922) considers various index number formulas, but he does not give any exact definition of the concept. The same is true for Frisch (1930), Wald (1937), Swamy (1965) and Eichhorn (1976), perhaps the most authorative investigators of Fisher's test approach. There does not exist any generally accepted definition of the concept of an index formula in the descriptive (statistical, atomistic) approach. As we found out in the economic approach there are several exact but competing definitions.

The index number formulas P_0^1 and Q_0^1 should be real valued functions of 4n-vectors (p^1,p^0,q^1,q^0) and (q^1,q^0,p^1,p^0) respectively having some characteristic properties. These properties are divided here into two groups, 'basic properties' required to be possessed by all index formulas and 'desiderata', which may or may not be satisfied. We propose and use the following definition.

Definition. Index number formula:

Let n be a positive integer and f a positive real valued function defined for all 4n-vectors (x^1,x^0,y^1,y^0) having positive components, $f:\mathbb{R}_+^{4n}\to\mathbb{R}_+$. Suppose that f has the following basic properties:

A. <u>The commodity reversal test</u>

$\forall\psi:\psi$ is a permutation of $(x_1,\ldots,x_n)$: $\forall(x^1,x^0,y^1,y^0)\in\mathbb{R}_+^{4n}$:

$f(\psi(x^1),\ \psi(x^0),\ \psi(y^1),\ \psi(y^0)) = f(x^1,x^0,y^1,y^0)$.

B. <u>The unit of measurement test</u>

$\forall\lambda\in\mathbb{R}_+^n$: $\forall(x^1,x^0,y^1,y^0)\in\mathbb{R}_+^{4n}$:

$$f((\lambda_1 x_1^1,\ldots,\lambda_n x_n^1),(\lambda_1 x_1^0,\ldots,\lambda_n x_n^0),(\frac{y_1^1}{\lambda_1},\ldots,\frac{y_n^1}{\lambda_n}),(\frac{y_1^0}{\lambda_1},\ldots,\frac{y_n^0}{\lambda_n})) =$$

$$= f(x^1,\ x^0,\ y^1,y^0)$$

C. <u>The monetary unit test</u>

$\forall c\in\mathbb{R}_+$: $\forall d\in\mathbb{R}_+$: $\forall(x^1,x^0,y^1,y^0)\in\mathbb{R}_+^{4n}$:

$f(cx^1,cx^0,dy^1,dy^0) = f(x^1,x^0,y^1,y^0)$.

D. <u>The weak proportionality test</u>

$\forall k\in\mathbb{R}_+$: $\forall(x^1,x^0,y^1,y^0)\in\mathbb{R}_+^{4n}$:

$f(kx^0,x^0,y^0,y^0) = k$.

E. <u>The weak identity test</u>

$\forall k\in\mathbb{R}_+$: $\forall(x^1,x^0,y^1,y^0)\in\mathbb{R}_+^{4n}$:

$f(x^0,x^0,ky^0,y^0) = 1$.

Then f is an <u>index number formula</u>.

In property A we used the concept of a permutation, which may be defined as follows: A linear mapping $\psi = (\psi_1,\ldots,\psi_n): \mathbb{R}^n \to \mathbb{R}^n$ is a permutation (of the components) of its argument vector $x = (x_1,\ldots,x_n)$ if in every column and row of its transformation matrix there is only one nonzero element which equals unity. This means that for all i, k and $x \in \mathbb{R}^n$, $D_i\psi_k(x) = 0$ or 1 and $\sum_{i=1}^{n} D_i\psi_k(x) = 1$ for all k and $\sum_{k=1}^{n} D_i\psi_k(x) = 1$ for all i.

By giving different interpretations to the 4n-vector (x^1,x^0,y^1,y^0) we get price and volume index formulas.

A price index formula is a function f applied to a 4n-vector of the form (p^1,p^0,q^1,q^0) and mapping it to the number $f(p^1,p^0,q^1,q^0)$. This is a price index comparing the the price-quantity situation (p^1,q^1) to the situation (p^0,q^0). If $p^1 = kp^0$ and $q^1 = q^0$ we have according to D: $f(kp^0,p^0,q^0,q^0) = k$. We sometimes use the notation $f(p^1,p^0,q^1,q^0) = P(p^1,p^0,q^1,q^0)$, or shortly P_{t0}^{t1} or P_0^1 if the arguments are determined from the situation.

A quantity index formula is a function f applied to a 4n-vector of the form (q^1,q^0,p^1,p^0) and mapping it to the number $f(q^1,q^0,p^1,p^0)$. This is a quantity index comparing the price-quantity situation (p^1,q^1) to the situation (p^1,q^0). It has according to D the basic property of a quantity index: $f(kq^1,q^0,p^0,p^0) = k$. We use for the quantity index $f(q^1,q^0,p^1,p^0)$ sometimes alternative symbols $Q(q^1,q^0,p^1,p^0)$, Q_{t0}^{t1} or Q_0^1.

Our definition of index number formulas uses data from the two price-quantity situations (p^1,q^1), (p^0,q^0) only. Therefore a more complete name for our index number formula f would be a direct comparison (d.c.) index number formula. If we make an indirect price comparison of t_0 and t_1 via a third situation, say t_3, by defining $P_0^1 = P_0^3 P_3^1$, where P_0^3 and P_3^1 are calculated by direct comparison index number formulas then P_0^1 usually depends on (p^3, q^3). Therefore this P_0^1 is usually not a d.c. index number formula between situations t_0 and t_1. The two main strategies (i.e. base and chain methods) for producing comparisons between various price - quantity situations are discussed in chapter 4. The base and chain methods are examples of how d.c. index number formulas can be used in constructing index series. Of course more complicated strategies for constructing index series may be imagined and even strategies which cannot be represented using direct comparison index number formulas only.[1] We do not, however, discuss these more complicated strategies here.[2]

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1) Define P_{t-1}^t by $\log P_{t-1}^t = \sum \frac{1}{2}(w_i^{t-2} + w_i^{t-1}) \log (p_i^t / p_i^{t-1})$

This P_{t-1}^t is not a direct comparison index number formula, because it contains the value shares w_i^{t-2} from year t-2. It may offer, however, a reasonable comparison of prices between t-1 and t if the value shares change slowly. Chaining P_{t-1}^t:s provides an example of a strategy which cannot be represented using direct comparision index number formulas only. It seems to me very difficult to state in any general terms which is and which is not a reasonable strategy if strategies of this kind are included.

2) Note also that we have defined an index number formula f as a <u>function</u> from $\mathbb{R}_+^{4n}$ into $\mathbb{R}_+$. Any function f has many different representations, i.e. different "formulas", which determine the function. For instance Laspeyres' price index formula, $P_0^1 = \Sigma p_i^1 q_i^0 / \Sigma p_i^0 q_i^0$ or no.53 in Fisher's (1922) system, determines the same function $f: \mathbb{R}_+^{4n} \to \mathbb{R}_+$ as the following formulas:

Fisher's no.	"formula" or the method of calculation
3	$\Sigma p_i^0 q_i^0 (p_i^1/p_i^0) / \Sigma p_i^0 q_i^0$
6	$(V^1/V^0) / (\Sigma q_i^0 p_i^1 (q_i^1/q_i^0) / \Sigma q_i^0 p_i^1$
17	$\Sigma p_i^1 q_i^0 / \Sigma p_i^1 q_i^0 (p_i^0/p_i^1)$
20	$(V^1/V^0) / (\Sigma q_i^1 p_i^1 / \Sigma q_i^1 p_i^1 (q_i^0/q_i^1))$
60	$(V^1/V^0) / (\Sigma q_i^1 p_i^1 / \Sigma q_i^0 p_i^1)$

Still other expressions as $\Sigma w_i^0 (p_i^1/p_i^0)$, $p^1 \cdot q^0 / p^0 \cdot q^0$, etc. are easily found.

Perhaps the choice of our conditions can be motivated best in the way Fisher (1922) originally did. The commodity reversal test was introduced by Fisher p. 63 as follows

> "In short, we must, in some sense, treat alike: (a) any two commodities; (b) the two times; (c) the two factors.
> The first test is seldom if ever violated. It is mentioned here for completeness and to afford a basis for a better appreciation of the two less obvious tests which follow. In order to avoid confusion the three tests will be distinguished as:
> "Preliminary" - The commodity reversal test
> Test 1 - The time reversal test
> Test 2 - The factor reversal test
>
> Any formula to be fair should satisfy all three tests. The requirement as to commodities is that the order of the commodities ought to make no difference - that, to be specific, any two commodities could be interchanged, i.e. their order reversed, without affecting the resulting index number. This is so simple as never to have been formulated. It is merely taken for granted and observed instinctively. Any rule for averaging the commodities must be so general as to apply interchangeably to all of the terms averaged. It would not be fair, for instance, arbitrarily to average the first half of the commodities by the arithmetic method and the other half by the geometric, nor fancifully to weight the seventh commodity by 7 and the tenth commodity by 10 so that if the seventh and tenth commodities were interchanged the result would be affected."

The time and factor reversal test will be commented on later when they are introduced into our system as desiderata. The commodity reversal test rules out formulas such as $P_0^1 = 0.2(p_1^1/p_1^0) + 0.8(p_2^1/p_2^0)$ or $P_0^1 = (2p_1^1 + 8p_2^1)/(2p_1^0 + 8p_2^0)$. But does it exclude the formula of Lowe (which is no. 9051 in Fisher's system) as well,

$$(10) \qquad P_0^1 = p^1 \cdot q/p^0 \cdot q \ ,$$

where $q = (q_1,\ldots,q_n) \in \mathbb{R}_+^n$ is a fixed vector describing quantities of $a_1,\ldots,a_n$, which q is independent of q^0 and q^1. A common sense calculator of Lowe's index (10) of course permutes q together with p^0 and p^1 to get

(11) $\psi(p^1)\cdot\psi(q)/\psi(p^0)\cdot\psi(q)$,

which necessarily equals (10). So (10) seems to satisfy the commodity reversal test. However, we must formulate (10) according to our definition of an index formula and a straight interpretation would be a function f such that

(12) $\exists q\in\mathbb{R}^n_+ : \forall\, (p^1.p^0,q^1,q^0)\in\mathbb{R}^{4n}_+ :$

$$f(p^1,p^0,q^1,q^0) = p^1\cdot q/p^0\cdot q \quad .$$

But here f does not satisfy A, because q is not permuted although p^1,p^0,q^1 and q^0 were. The interpretation (12) describes the behaviour of an unsensible calculator (e.g. electronic computer) of Lowe's index (10). But is there any interpretation of (10) as a function $(p^1,p^0,q^1,q^0) \to f(p^1,p^0,q^1,q^0)$ which satisfies (13)-(15):

(13) $f(p^1,p^0,q^1,q^0) = p^1\cdot q/p^0\cdot q$

(14) $f(\psi(p^1),\psi(p^0),\psi(q^1),\psi(q^0)) = \psi(p^1)\cdot\psi(q)/\psi(p^0)\cdot\psi(q)$

(15) $p^1\cdot q/p^0\cdot q = \psi(p^1)\cdot\psi(q)/\psi(p^0)\cdot\psi(q)$,

where ψ is a permutation.

We prove that no such fixed function $f:\mathbb{R}^{4n}_+\to\mathbb{R}_+$ exists. Equation (15) is always trivially true. To see that (13) and (14) must be different functions calculate the first partial derivates $D_1\ f(x^1,x^0,y^1,y^0)$ for $x^1=x^0=y^1=y^0=\vec{1}$ (a vector of unities) from

(13) and (14). We get $q_1/\Sigma q_i$ and $\psi_1(q)/\Sigma\psi_i(q) = \psi_1(q)/\Sigma q_i$ respectively. These are not generally equal, unless $q_1=\ldots=q_n$

when (10) reduces to $\Sigma p_i^1/\Sigma p_i^0$. Therefore (10) does not satisfy the commodity reversal test unless all q_i:s are equal, in which case it does not satisfy the unit of measurement test. I have found nowhere in Fisher (1922) a statement that all his formulas 9001-9051 fail to satisfy the commodity reversal test unless the weights are equal.

The essential point in the discussion was that q was regarded as independent of q^1 and q^0. If q is an approximation to, say, their average $\frac{1}{2}(q^1+q^0)$ the situation changes qualitatively.

Let us again use (10) but define now

$$(16) \qquad q_i = h(q_i^1, q_i^0) = 10^k \ \mathrm{INT}[\tfrac{1}{2}(q_i^1+q_i^0)10^{-k}] ,$$

where k is the unique integer for which $\frac{1}{2}(q_i^1+q_i^0)10^{-k} \in [10, 100)$ and INT(x) is the greatest integer m for which $m \leq x$.
Now $q_i \leq \frac{1}{2}(q_i^1+q_i^0)$ but q_i falls short of $\frac{1}{2}(q_i^1+q_i^0)$ by less than 10 % and its values are integers 10,11,...,99 multiplied by 10^k.
So the representation of this P_0^1 reads

$$(17) \qquad f(p^1,p^0,q^1,q^0) = \Sigma p_i^1 h(q_i^1,q_i^0)/\Sigma p_i^0 h(q_i^1,q_i^0) .$$

This f satisfies A but is not a continuous function of q^1 and q^0. Partly for this reason we did not include continuity in our list of basic properties. However, although (17) almost satisfies the unit of measurement test it does not satisfy it exactly. Change, for instance, only the unit of a_1 so that q_1^k changes to q_i^k/λ and p_1^k to λp_1^k (k = 0,1). The only terms affected are

$$(18) \qquad p_1^k h(q_1^1,q_1^0) \to \lambda p_1^k h(q_1^1/\lambda,\ q_1^0/\lambda) \quad , \quad k = 0,1 .$$

Unfortunately these depend on λ, because λp_1^k increases continuously with λ but $h(q_1^1/\lambda,\ q_1^0/\lambda)$ decreases only stepwise. Therefore (17) does not qualify for a price index in our system. Anyhow (17) is always a good approximation to a good price index formula

$$(19) \qquad P_0^1 = p^1 \cdot (q^1+q^0)/p^0 \cdot (q^1+q^0) \quad .$$

We thus propose to regard (17) as a good approximation to a good index formula, not as a good index formula. These complications should show how strong such qualitative properties as A and B prove to be.

The unit of measurement test is presented in Fisher (1922) p. 420 under the head of commensurability:

> "An index number of prices should be unaffected by changing any unit of measurement of prices and quantities.
>
> This test eliminates all the "ratios of averages" as shown in Appendix III and also Formula 51 in our numbered series, together with those derived from, or dependent on 51, viz. 52 and 521. All the other formulae obey this test, which may be considered of fundamental importance in the theory of index numbers."

Here Fisher fails to note that Lowe's formula (10), i.e. his formula 9051, does not satisfy the unit of measurement test.

Examples of the "ratios of averages" considered by Fisher on p. 451-457 include, e.g., the simple averages $\frac{1}{n}\Sigma p_i^1/\frac{1}{n}\Sigma p_i^0 = \Sigma p_i^1/\Sigma p_i^0$,

$$\Sigma(1/p_i^0)/\Sigma(1/p_i^1) \ , \ (\Pi p_i^1)^{1/n}/(\Pi p_i^0)^{1/n} = (\Pi(p_i^1/p_i^0))^{1/n} \ ,$$

of which the last is an "average of ratios" as well and thus qualifies (in this respect) for a price index formula.

Of the weighted averages, e.g.,

$$(20) \qquad \frac{\Sigma q_i^1 p_i^1}{\Sigma q_i^1} \Big/ \frac{\Sigma q_i^0 p_i^0}{\Sigma q_i^0} = \frac{\Sigma q_i^1 p_i^1}{\Sigma q_i^0 p_i^0} \Big/ \frac{\Sigma q_i^1}{\Sigma q_i^0} \ ,$$

does not satisfy this test but in

$$(21) \qquad \frac{\Sigma q_i^0 p_i^1}{\Sigma q_i^0} \Big/ \frac{\Sigma q_i^0 p_i^0}{\Sigma q_i^0} = \frac{\Sigma q_i^0 p_i^1}{\Sigma q_i^0 p_i^0}$$

the sum of old quantities cancels out and this removes all traces of incommensurability, Fisher (1922) p. 455. Fisher discusses on p. 456 the cases where averages of prices (e.g., 20) can properly be used:

> "The only cases in which it is really justifiable to use the genuine method of taking the ratio of averages is where the units are really or nearly commensurable. Thus, it is entirely legitimate to obtain the index number of various quotations of one special kind of commodity, such as salt, by taking the average of its prices in different markets. In such a case the precaution, so essential in the previous examples, of forcibly altering numerator to suit denominator, or vice versa, does not need to be taken. The true average for each year can be taken independently of the other years. Another case is where the commodities are of one general group, such as kinds of coffee or fuels, e.g. coal and coke where the same unit, such as the ton, is used for all so that there is no danger of changing one without, at the same time, changing the others equally.
>
> The most interesting practical examples, however, are the average wage of different but similar kinds of labor and the average price of different but similar kinds of securities, in which cases the objection of incommensurability applies but not very strongly. In the stock market the average price of stocks is taken, the "common unit", if it may be so called, being the 'par value'."

Fisher thus argues that, e.g., (20) is a proper index number formula if "the units are really or nearly commensurable". This is the central problem of the quality changes, which cannot be pushed aside as easily as Fisher does. We cannot approve (20) as a proper price index (in our axiomatic system), but it may offer a good approximation to proper price indices under some circumstances. Without attempting a satisfactory treatment of these most interesting and difficult problems we want to make the following remarks. Let us imagine that the commodities $a_1,\ldots,a_n$ for which the 'price index' (20) has been calculated are various kinds of coke of, say, SITC subgroup 321.8 as in our imports example in chapter 7. Their quantities $q_1,\ldots,q_n$ are measured in tons, so that (20) is the ratio of new and old average unit values per ton or, equivalently, the value ratio divided by total new and old quantities expressed in tons. If (20) is accepted as a price index, this means that the ratio of the new and old total quantities in tons,

(22) $$\Sigma q_i^1/\Sigma q_i^0 ,$$

should be accepted as a proper quantity index corresponding to it. It is easier and more natural to discuss the merits and dismerits of the 'quantity index' (22) than those of the 'price index' (20). If coke were perfectly homogeneous material then there would not exist any aggregation problems and (22) would be the proper quantity relative for coke. If the prices of these 'various kinds' of coke, $a_1,\ldots,a_n$, are identical, $p_i^1 = p_j^1$ and $p_i^0 = p_j^0$ for all i and j, then there is really no _theoretical_ reasons for the separation of $a_1,\ldots,a_n$, although it may arise from _institutional_ or other _practical_ reasons. In this case we have

(23) $$\Sigma q_i^1/\Sigma q_i^0 = \Sigma q_i^1 p_i^1/\Sigma q_i^0 p_i^1 = q^1\cdot p^1/q^0\cdot p^1$$

because $p_i^1 = p_j^1$ for all i and j and similarly

(24) $$\Sigma q_i^1/\Sigma q_i^0 = \Sigma q_i^1 p_i^0/\Sigma q_i^0 p_i^0 = q^1\cdot p^0/q^0\cdot p^0$$

if $p_i^0 = p_j^0$ for all i and j. Thus if various kinds of coke are perfectly homogeneous in their quality[1] which should be reflected in such a way that their prices are equal, then (22) would equal both Paasche's and Laspeyres' quantity indices (or in fact any reasonable quantity index: we have no index problem in this case).

If the old (or new) prices for various kinds of coke differ only little because of minor quality changes or for other reasons, then (23) or (24) are true only approximately. Their right hand sides continue to be proper quantity indices, but their calculation would be actually futile, because (22) would give practically the same results. The situation changes drastically if the units of measurement of some a_i:s are changed to say kilograms and others are kept unchanged. Then the terms of (22) are empirically meaningless sums of numbers expressing quantities of a_i in tons or kilograms. Equations (23) and (24) are no longer even approximately true, while their right hand sides remain unchanged because Paasche's and Laspeyres' indices satisfy the unit of measurement test.

---

1) We do not discuss the problems caused by, e.g., regional differences in prices.

Our analysis is completely different to that of Samuelson and Swamy (1974) p. 571, who claim that "the literature, from Fisher on, including Samuelson (1967, p. 25) and Swamy[1)] (1965, p. 620), is inadequate on the dimensional invariance test". They appeal to dimensional analysis and require that appropriate dimensional constants be added to price and quantity indices. But this means that, if the units of measurement in the variables p^1, p^0, q^1, q^0 are changed, the index function f should be changed accordingly. As they admit, "once one has introduced the appropriate dimensional constants, we impose thereby no restrictions on the functional form of the index number." Their analysis (containing a bad misprint in the example they give) is in my opinion unsatisfactory. Why not write the 'dimensional constants' explicitly in the functional form of the index number, so as to get a function satisfying our unit of measurement test? What is the use of their 'test' which is satisfied by any index number formula?

Thus we see that great precision is needed in formulating these 'common sense' requirements. We otherwise easily find 'common sense' in connections where it is apparently lacking, e.g. unsensible use of (22) is either approved or ruled out from considerations as Samuelson and Swamy (1974) do.

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1) Swamy (1965) considers only the case $\lambda_1=\ldots=\lambda_n$ as Eichhorn (1976) has also noted, this being in my opinion the inadequacy of Swamy's treatment.

The monetary unit test is here expressed as a symmetrical property although a change in the monetary unit in fact changes only prices while quantities remain unaffected. The natural interpretation of C is in the case of a price index $P(cp^1,cp^0,q^1,q^0) = P(p^1,p^0,q^1,q^0)$ and in the case of a quantity index $Q(q^1,q^0,dp^1,dp^0) = Q(q^1,q^0,p^1,p^0)$. Both these special cases of C could have been represented (in our axiomatics, which includes the unit of measurement test B) in the following equivalent form

C' $\forall m \in \mathbb{R}_+ : \forall (x^1,x^0,y^1,y^0) \in \mathbb{R}_+^{4n} : f(mx^1,mx^0,y^1,y^0) = f(x^1,x^0,y^1,y^0)$.

For, according to B, by setting $\lambda_1 = \ldots = \lambda_n = 1/d$ we have

(25) $$f(x^1/d,x^0/d,dy^1,dy^0) = f(x^1,x^0,y^1,y^0)$$

and using C' and inserting m = cd we get

(26) $$f(cx^1,cx^0,dy^1,dy^0) = f(x^1/d,x^0/d,dy^1,dy^0) = f(x^1,x^0,y^1,y^0) .$$

This implies our original C.

We have chosen very weak forms of the proportionality and identity tests for our axiomatics. Usually stronger forms are introduced; e.g., Fisher (1922) p. 420 formulates his proportionality test as follows:

> "An index number of prices should agree with the price relatives if those agree with each other."
> "The test of proportionality is really a definition of an average. It is fulfilled among the primary formulae by all the odd numbered formulae. But none of the even numbered formulae fulfill it (except Laspeyres' and Paasche's, which are also odd numbered)."

Fisher's proportionality test is much stronger than ours as it requires that $f(kx^0,x^0,y^1,y^0) = k$ for all y^1 and y^0. These other proportionality tests will be presented here as desiderata.

The identity test is not explicitly presented in Fisher (1922), but it follows in a strong form

$$(27) \qquad \forall (x^1,x^0,y^1,y^0)\in \mathbb{R}_+^{4n} : f(x^0,x^0,y^1,y^0) = 1$$

from Fisher's proportionality test. The identity test is explicitly mentioned by Frisch (1936) as $P_0^0 = 1$ (in our notation), which obviously means a very weak form

$$(28) \qquad \forall (x^1,x^0,y^1,y^0)\in \mathbb{R}_+^{4n} : f(x^0,x^0,y^0,y^0) = 1 .$$

This is implied, e.g., by our proportionality test D. Our formulation E is slightly stronger and it will guarantee some useful properties for the collection F of all index number formulas f. Both D and E are implied by the following proportionality test[1)]

$$(29) \qquad \forall k\in \mathbb{R}^+ : \forall m\in \mathbb{R}^+ : \forall (x^1,x^0,y^1,y^0)\in \mathbb{R}_+^{4n} : f(kx^0,x^0,my^0,y^0) = k ,$$

which we shall, however, present as a desideratum. Note that even the identify test (28) is violated by some functions regarded sometimes as index number formulas. Calculate the price change $\bar{P}_0^1$ via a third point as $P_0^3 \cdot P_3^1$ using Paasche's index: $P_0^1 = (p^3 \cdot q^3 / p^0 \cdot q^3) \cdot (p^1 \cdot q^1 / p^3 \cdot q^1)$. This is not equal to one although $p^0 = p^1$ and $q^1 = q^0$ as required by (28). In the same way we can show that, e.g., the chain index $\bar{P}_0^2 = P_0^1 P_1^2$ need not be unity although $p^2 = p^0$ and $q^2 = q^0$.

---

1) This is proposed by Y. Vartia (1976a). Eichhorn (1976) p. 255 mentioned a similar proportionality test, where in addition k = m.

Next we investigate some closure properties of our collection F of all index number formulas satisfying our definition. Let $f \in F$. Then $f(p^1,p^0,q^1,q^0)=P_0^1$ is a price index formula. If prices and quantities are interchanged we get a quantity index $Q_0^1=f(q^1,q^0,p^1,p^0)$ calculated from the same formula. We have formulated our axions so as to apply both to price and quantity index formulas. Therefore, e.g., the monetary unit test was formulated symmetrically in x:s and y:s.

In fact our system of indices is constructed so that any of our formulas $f \in F$ is applicable both as a price and a quantity index formula. The 'important arguments' in $f(x^1,x^0,y^1,y^0)$ are the first ones, here the x:s, which determine the main properties of the index number formula. If we take an $f \in F$ and define a function $\bar{f}:\mathbb{R}_+^{4n} \to \mathbb{R}_+$ such that $\bar{f}(x^1,x^0,y^1,y^0) = f(y^1,y^0,x^1,x^0)$, then $\bar{f} \notin F$. This function $\bar{f}$ is <u>not</u> an index number formula, because $\bar{f}(kx^0,x^0,y^0,y^0) = 1$ and not k and $\bar{f}(x^0,x^0,my^0,y^0) = k$ and not 1 as they should be according to D and E.

But if we divide the value ratio $p^1 \cdot q^1/p^0 \cdot q^0$ by the quantity index $Q_0^1 = f(q^1,q^0,p^1,p^0)$ we get a new function

$$(30) \qquad \bar{f}(p^1,p^0,q^1,q^0) = (p^1 \cdot q^1/p^0 \cdot q^0)/f(q^1,q^0,p^1,p^0)$$

which should qualify for a price index formula. Fisher (1922) p. 125 calls this formula the <u>factor antithesis</u> of f. The procedure (30) applies to quantities in the same way as to prices and the same function $\bar{f}$ is defined.

An essential feature in our axiomatics is that F is closed under the operation of calculating (defining) a factor antithesis of a formula:

Theorem 1. Let $f \in F$ and define a function $\bar{f}: \mathbb{R}_+^{4n} \to \mathbb{R}_+$ as follows:

$$\forall (x^1, x^0, y^1, y^0) \in \mathbb{R}_+^{4n}: \; \bar{f}(x^1, x^0, y^1, y^0) = \frac{x^1 \cdot y^1 / x^0 \cdot y^0}{f(y^1, y^0, x^1, x^0)} \quad (31)$$

Then $\bar{f} \in F$. (This index number formula $\bar{f}$ is the factor antithesis of f.)

Proof: As already stated, $\bar{f}$ is defined in $\mathbb{R}_+^{4n}$ and is a real valued and positive function, because it is a ratio of such functions. We have to show that $\bar{f}$ satisfies the properties A-E.

A. $x^1 \cdot y^1 / x^0 \cdot y^0$ and $f(y^1, y^0, x^1, x^0)$ satisfy the commodity reversal test and thus their ratio is independent of the permutation of the argument vectors x^1, x^0, y^1, y^0.

B-C. A change of units leaves $x^1 \cdot y^1 / x^0 \cdot y^0$, $f(y^1, y^0, x^1, x^0)$ and, thus, their ratio invariant.

D. Let $k \in \mathbb{R}_+$ and $(x^1, x^0, y^1, y^0) \in \mathbb{R}_+^{4n}$. Then

$$\bar{f}(kx^0, x^0, y^0, y^0) = (kx^0 \cdot y^0 / x^0 \cdot y^0) / f(y^0, y^0, kx^0, x^0)$$
$$= k / f(y^0, y^0, kx^0, x^0).$$

We see that $\bar{f}(kx^0, x^0, y^0, y^0) = k$ if and only if $f(y^0, y^0, kx^0, x^0) = 1$.

(This is precisely the reason why we have added the weak identity test E to the basic properties, cf. Y. Vartia (1974).) The property E is the weakest identity test which guarantees that $\bar{f}$ satisfies D.

E. Let $k \in \mathbb{R}_+$ and $(x^1,x^0,y^1,y^0) \in \mathbb{R}^{4n}_+$. Then

$$\bar{f}(x^0,x^0,ky^0,y^0) = (kx^0 \cdot y^0/x^0 \cdot y^0)/f(ky^0,y^0,x^0,x^0)$$
$$= k / k = 1,$$

because $f(ky^0,y^0,x^0,x^0) = k$ according to E. □

<u>Corollary</u>: The weak factor reversal test.

$$\forall f \in F: \exists \bar{f} \in F: \forall (x^1,x^0,y^1,y^0) \in \mathbb{R}^{4n}_+:$$
$$f(x^1,x^0,y^1,y^0)\bar{f}(y^1,y^0,x^1,x^0) = x^1 \cdot y^1/x^0 \cdot y^0.$$

This (or something similar to it) is sometimes called the product test, see Törnqvist (1974) and Eichhorn (1976). Usually the meaning of the weak factor reversal test as a test or a criterion for index numbers is unclear. For instance, Diewert (1976b, p. 115) uses the equation $P_0^1 Q_0^1 = V^1/V^0$ merely as a means of defining a quantity index (or just a function) Q_0^1 when P_0^1 is given. This kind of equation is not a <u>test</u>. We have formulated the weak factor reversal test as a test for the collection F of index number formulas.

If we change the time periods in a price index formula $P_0^1 = f(p^1,p^0,q^1,q^0)$ we get a price index $P_1^0 = f(p^0,p^1,q^0,q^1)$ measuring the relative price change from t_1 to t_0. By taking the reciprocal of this number, $1/P_1^0 = 1/f(p^0,p^1,q^1,q^0)$, a new function $\bar{f}$ is defined

(32) $\quad \bar{f}(p^1,p^0,q^1,q^0) = 1/f(p^0,p^1,q^1,q^0),$

which should qualify for a price index formula measuring the relative price change from t_0 to t_1. This function $\bar{f}$ is called by Fisher (1922) p. 118 the time antithesis of the original formula f. The procedure applies to quantities as well. In our axiomatics the time antithesis of any formula $f \in F$ qualifies for an index number formula.

Theorem 2. Let $f \in F$ and define a function $\bar{f}$: $\mathbb{R}_+^{4n} \to \mathbb{R}_+$ as follows:

(33) $\quad \forall (x^1,x^0,y^1,y^0) \in \mathbb{R}_+^{4n}$: $\bar{f}(x^1,x^0,y^1,y^0) = 1/f(x^0,x^1,y^0,y^1)$.

Then $\bar{f} \in F$. (This index number formula $\bar{f}$ is the time antithesis of $\bar{\bar{f}}$.)

Proof: There is no difficulty to show that $\bar{f}$ satisfies the properties A-C.

D. Let $k \in \mathbb{R}_+$ and $(x^1,x^0,y^1,y^0) \in \mathbb{R}_+^{4n}$. Then

$$\begin{aligned}\bar{f}(kx^0,x^0,y^0,y^0) &= 1/f(x^0,kx^0,y^0,y^0)\\ &= 1/f(\tfrac{1}{k}z^0,z^0,y^0,y^0), \text{ where } z^0 = kx^0\\ &= 1/(\tfrac{1}{k}), \text{ from D}\\ &= k\end{aligned}$$

The proof for property E is similar. □

Our theorems 1 and 2 allow us to use the time or factor antithesis of any of our formulas $f \in F$ because these are index number formulas in our system. The following theorem shows that every mean $K(f_1,f_2)$ of index number formulas f_1 and f_2 is itself an index number formula:

Theorem 3. Let $f_1 \in F$ and $f_2 \in F$. Then $K(f_1,f_2) \in F$ where $K(x,y)$ is a mean of positive real numbers x and y.

Proof: A mean $K(x,y)$ of positive real numbers is a function $K: \mathbb{R}_+^2 \to \mathbb{R}_+$ having the properties

A. $\forall (x,y) \in \mathbb{R}_+^2$: $\min(x,y) \leq K(x,y) \leq \max(x,y)$.

B. K is a continuous function

C. $\forall a \in \mathbb{R}_+$: $\forall (x,y) \in \mathbb{R}_+^2$: $K(ax,ay) = aK(x,y)$.

D. $\forall (x,y) \in \mathbb{R}_+^2$: $K(x,y) = K(y,x)$.

Let f_1 and f_2 be index number formulas and K any mean defined (at least) for positive x and y. Let $k \in \mathbb{R}_+$ and $(x^1,x^0,y^1,y^0) \in \mathbb{R}_+^{4n}$. Then the tests A–C are clearly satisfied. The test D is satisfied because, if $f_1(kx^0,x^0,y^0,y^0) = k$ and $f_2(kx^0,x^0,y^0,y^0) = k$, then their mean $K(f_1,f_2) = k$ because of A. The proof for the weak identity test is similar. □

Fisher uses particularly the geometric mean $G(x,y) = \sqrt{xy}$ to rectify his formulas f with their time or factor antitheses, see Fisher (1922) p. 136. The reason for using the geometric mean in averaging, instead of, e.g., the arithmetic or harmonic mean, is that the geometric mean of f and $\bar{f}$, which are time (factor) antitheses of each other, always satisfies the time (factor) reversal test.

We proceed to formulate these and other desiderata which a good index number formula ought to satisfy.

3.3. Desiderata concerning index number formulas

Desideratum 1. The time reversal test[1]:

(34) $$\forall(x^1,x^0,y^1,y^0) \in \mathbb{R}_+^{4n}: f(x^0,x^1,y^0,y^1) = 1/f(x^1,x^0,y^1,y^0).$$

This means that if the time periods are interchanged the index changes into its resiprocal, which is a property of price and quantity relatives. In other words an index number formula f ought to equal its time antithesis $\bar{f}$. If the periods are treated asymmetrically as, e.g., in Laspeyres' price index formula

(35) $$f(p^1,p^0,q^1,q^0) = p^1 \cdot q^0 / p^0 \cdot q^0$$

the time reversal test is not satisfied.

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1) Also called the point reversal test, Frisch (1936); and the inversion criterion, Törnqvist (1935, 1974).

Desideratum 2. The factor reversal test:

(36) $$\forall(x^1,x^0,y^1,y^0) \in \mathbb{R}_+^{4n}: f(x^1,x^0,y^1,y^0) \cdot f(y^1,y^0,x^1,x^0) = x^1 \cdot y^1 / x^0 \cdot y^0.$$

This corresponds to Törnqvist's (1974) multiplication and symmetry criteria. Written in prices and quantities this reads

(37) $$f(p^1,p^0,q^1,q^0) f(q^1,q^0,p^1,p^0) = p^1 \cdot q^1 / p^0 \cdot q^0.$$

Here $f(p^1,p^0,q^1,q^0) = P_0^1$ is a price index calculated from formula f, and $f(q^1,q^0,p^1,p^0) = Q_0^1$ is a quantity index calculated from the same formula f. The factor reversal test says that the product of these two, $P_0^1 Q_0^1$, should be the value ratio $p^1 \cdot q^1 / p^0 \cdot q^0$. In other words an index number formula f should equal its factor antithesis $\bar{f}$.

Fisher's method of 'rectifying' formulas is based on the following theorems.

Theorem 4. Let $f_1 \in F$ and $\bar{f}_1$ be its time antithesis. Then $\sqrt{f_1 \bar{f}_1} \in F$ and satisfies the time reversal test.

Proof: $\bar{f}_1 \in F$ by theorem 2 and $\sqrt{f_1 \bar{f}_1} \in F$ by theorem 3. Denote $f_2 = \sqrt{f_1 \bar{f}_1}$.

Then $f_2(x^1,x^0,y^1,y^0) = [f_1(x^1,x^0,y^1,y^0)/f_1(x^0,x^1,y^0,y^1)]^{\frac{1}{2}}$.

The time antithesis of f_2 is by definition

$$\begin{aligned}\bar{f}_2(x^1,x^0,y^1,y^0) &= 1/f_2(x^0,x^1,y^0,y^1)\\ &= 1/[f_1(x^0,x^1,y^0,y^1)/f_1(x^1,x^0,y^1,y^0)]^{\frac{1}{2}}\\ &= f_2(x^1,x^0,y^1,y^0)\end{aligned}$$

Therefore $\bar{f}_2$, the time antithesis of f_2, equals f_2, which means that f_2 satisfies the time reversal test. □

Theorem 5. Let $f_1 \in F$ and $\bar{f}_1$ be its factor antithesis. Then $\sqrt{f_1\bar{f}_1} \in F$ and satisfies the factor reversal test. Theorem 5 is proved in the same way as theorem 4.

The best index number formulas - e.g., Fisher's ideal index, the index suggested by Stuvel (1957) and our new indices - satisfy both time and factor reversal tests.

In the case of the next desideratum we have several alternatives. We have already included the weak proportionality test D in our definition of the index number formula. Here we add some stronger formulations of this test, which Fisher (1922) p. 420 states in the case of price index as follows: "An index number of prices should agree with the price relatives if those agree with each other".

<u>Desiderata 3.</u> Proportionality tests:

PT1: $f(kx^0,x^0,y^0,y^0) = k$

PT2: $f(kx^0,x^0,my^0,y^0) = k$

PT3: $f(kx^0,x^0,y^1,y^0) = k$

PT4: $\dfrac{f(kx^1,x^0,y^0,y^0)}{f(x^1,x^0,y^0,y^0)} = k$

PT5: $\dfrac{f(kx^1,x^0,my^0,y^0)}{f(x^1,x^0,y^0,y^0)} = k$

PT6: $\dfrac{f(kx^1,x^0,y^1,y^0)}{f(x^1,x^0,y^1,y^0)} = k$

PT7: $\dfrac{f(kx^1,x^0,my^1,y^0)}{f(x^1,x^0,y^1,y^0)} = k$

PT8: $\forall y^2 \in \mathbb{R}^n_+$: $\dfrac{f(kx^1,x^0,y^2,y^0)}{f(x^1,x^0,y^1,y^0)} = k$

These statements shall be valid for all positive k and m and for all $(x^1,x^0,y^1,y^0) \in \mathbb{R}^{4n}_+$.

These are only some of the most interesting possibilities. Here PT1 is our weak proportionality test D. PT2 implies PT1 and the weak identity test E in our definition of the index number formula. It is easily seen that PT3⇒PT2⇒PT1 and PT7⇒PT6⇒PT5⇒PT4⇒PT1. Also, PT5 implies PT2 but not PT3 as may be shown by counter-examples. Neither does PT4 imply PT3. We shall not investigate these interesting problems here but only point out, without proofs, that Laspeyres' formula satisfies all these proportionality criteria, Paasche's, Fisher's and Vartia II formulas satisfy all of them up to PT7, Stuvel's formula satisfies PT1-PT3 but Vartia Index I satisfies only PT1 and PT2, see Y. Vartia (1976a). Note that, e.g., Törnqvist (1973, 1974) gave the very strong version PT 8.

The Economic Quantity Index $Q(q^1,q^0; p^*)$ defined in chapter 2 satisfies $Q(kq^0,q^0; p^*) \equiv k$ only in the homothetic case when prices and quantities change proportionately together. This corresponds to PT2 as written for quantity indices:

$$(38) \qquad \forall k \in \mathbb{R}_+: \forall m \in \mathbb{R}_+: \forall (q^1,q^0,p^1,p^0) \in \mathbb{R}_+^{4n}:$$

$$q^1 = kq^0 \ \& \ p^1 = mp^0 \Rightarrow Q_0^1 = k.$$

If PT3 is interpreted to mean that $Q(kq^0, q^0; p^*) = k$ for any two equilibrium situations (p^0,q^0) and (p^1,kq^0), then it is not satisfied, see Samuelson and Swamy (1974). The Economic Price Index $P(p^1,p^0; q^*)$ always satisfies even $P(kp^1,p^0; q^*) = kP(p^1,p^0; q^*)$, which corresponds to PT8.

Taking two price indices P_0^2 and P_0^1 and dividing them we get a comparison between the prices p^2 and p^1: $\bar{P}_1^2 = P_0^2/P_0^1$. The base test says that this should be independent of the data from the base period t_0, see Eichhorn (1976). In addition, however, we impose the natural requirement that P_0^2/P_0^1 should be a price index formula:

<u>Desideratum 4.</u> The base test:

Let $f \in F$. This f satisfies the base test if and only if

(39) $\exists \bar{f} \in F$: $\forall (x^0,y^0) \in \mathbb{R}_+^{2n}$: $\forall (x^1,y^1) \in \mathbb{R}_+^{2n}$: $\forall (x^2,y^2) \in \mathbb{R}_+^{2n}$:

$$f(x^2,x^0,y^2,y^0)/f(x^1,x^0,y^1,y^0) = \bar{f}(x^2,x^1,y^2,y^1).$$

This is a very strong requirement, which , e.g., implies by inserting $y^2=y^1$ and $x^2=kx^1$

(40) $$\frac{f(kx^1,x^0,y^1,y^0)}{f(x^1,x^0,y^1,y^0)} = k$$

or PT6 for f. The best descriptive index number formulas do not satisfy this test. In practise formulas using constant weights, such as $P_0^1 = p^1 \cdot q/p^0 \cdot q$ and $\log P_0^1 = \Sigma c_i \log(p_i^1/p_i^0)$, $\Sigma c_i=1$, seem to be of the type satisfying (39). As we have shown, however, they do not satisfy the unit of measurement and commodity reversal tests and are therefore no d.c. index number formulas.

The base test as well as all the proportionality tests are weaker than the next desideratum , which especially some index theorists starting from the economic approach regard as highly important, see Samuelson and Swamy (1974) p. 575-6. In their opinion "the circular test is as required as is the property of transitivity itself". Fisher, however, dropped this circular property from his list of desirable properties of index numbers, as already stated in chapter 2.

Desideratum 5. The circular test:

(41) $\forall(x^0,y^0)\in\mathbb{R}_+^{2n}$: $\forall(x^1,y^1)\in\mathbb{R}_+^{2n}$: $\forall(x^2,y^2)\in\mathbb{R}_+^{2n}$:

$$f(x^2,x^0,y^2,y^0) = f(x^1,x^0,y^1,y^0) \cdot f(x^2,x^1,y^2,y^1).$$

This test is sometimes called the chain criterion and presented simply as $P_0^2 = P_0^1P_1^2$ for price indices.

The best index number formulas fail to meet this desideratum, this being because the shares $w_i(t)$ will change over time. To circumwent this problem the chain method to be described in the next chapter may be used. The suggested solution is that the direct comparison $t_0 \rightarrow t_2$ should not be made at all but, instead, the price change $\bar{P}_0^2$ should be computed with the aid of the partial component comparisons P_0^1 and P_1^2, by chaining the price indices and defining $\bar{P}_0^2 = P_0^1P_1^2$. This works in time series where we have a definite ordering between the periods. The length of the consecutive periods has, however, an effect on the results.

This solution is at variance with the one suggested by Fisher, who was of the opinion that a cost-of-living index, for instance, should always be computed using a base method: the cost of living in any one year should invariably be compared with the cost of living in the base year. As a matter of fact, this procedure is being applied in almost all countries of the world.

Yet, rather than being interested to know how the cost of living in a particular year relates to that of the base year we are usually interested in, for instance, the year-on-year changes in the cost of living. If the chain method is used, the movement of the cost of living will be determined precisely on the basis of these changes. It is to be expected that increasing acquaintance with the problems of index number calculations will be accompanied with an increasingly widespread use of the chain method. An elegant justification of the chain method is provided by Divisia-Törnqvist's integral formula to be presented in chapter 5.

The determinateness test is formulated by Fisher(1922) p. 420 as follows: "An index number of prices should not be rendered zero, infinity or undeterminate by an individual price becoming zero". Frisch (1936) formulated it analogously but demanded more: "... by an individual price or quantity becoming zero". Swamy (1965) p. 620 innocently demanded even more: "If any argument in $f(p^1,p^0,q^1,q^0)$ becomes zero or infinite, then f must not vanish, become infinite, or become indeterminate". (Here we have used our notation for the price index formula). A similar formulation was used by Samuelson and Swamy (1974) p. 572, but they "do not like this test" as Eichhorn (1976) notes. Swamy's requirement that even an infinite price or quantity should not affect P_0^1 badly seems inappropriate. E.g. Laspeyres', Paasche's and Fisher's indices, for instance, do not satisfy Swamy's requirement! Perhaps Swamy had the price or quantity relatives in mind? We adopt Eichhorn's (1976) rather weak formulation, which is stronger, however, than Fisher's original formulation.

Desideratum 6. The determinateness test:
If any scalar argument in $f(x^1,x^0,y^1,y^0)$ tends to zero, then $f(x^1,x^0,y^1,y^0)$ tends to a unique positive real number.

This is satisfied by, e.g., Laspeyres', Paasche's, Fisher's, Stuvel's and Vartia I and II indices, but Törnqvist's index does not satisfy it. The Economic Index Numbers also sometimes violate this criterion.

We should have at least two commodities in order to impose this criterion: its idea is that one (or a few!) exceptional commodities ought not influence the price index 'too much'. It is difficult to formulate this fully satisfactorily because it is a kind of common sense requirement, see Y. Vartia (1976a). We return to this problem in chapter 6 in a more concrete situation, where we investigate whether a formula reacts qualitatively correctly to extreme price and quantity changes.

Next the price index formula $f(p^1,p^0,q^1,q^0)$ will be written in another form, which reveals that it depens exclusively on the price and quantity ratios p_i^1/p_i^0, q_i^1/q_i^0 and the value shares $w_i^0 = v_i^0/V^0$. The same applies to the quantity index formula $f(q^1,q^0,p^1,p^0)$ because of symmetry.

It will first be noted that the 4n numbers

(42) $\quad p_i^1,\ p_i^0,\ q_i^1,\ q_i^0 \qquad i = 1,2,\ldots,n$

and the 4n+1 numbers

(43) $\quad p_i^1/p_i^0,\ q_i^1/q_i^0,\ w_i^0, q_i^0,\ V^0 \qquad i = 1,2,\ldots,n$

determine each other uniquely. In the latter set of data we need, in addition, e.g. the old total value $V^0 = \Sigma v_i^0$ as only n-1 of the value shares are mutually independent. We may thus write

(44) $\quad f(p^1,p^0,q^1,q^0) = \bar{g}(p_i^1/p_i^0, q_i^1/q_i^0, w_i^0, q_i^0, V^0 \mid i = 1,\ldots,n).$

By virtue of the unit of measurement test, $\bar{g}$ does not depend on the q_i^0:s, which may all be put equal to unity, say, without affecting the other arguments of $\bar{g}$. Furthermore, by the monetary unit test, $\bar{g}$ does not depend on V^0. It has thus been shown that any price index formula can always be written in the form

(45) $\quad f(p^1,p^0,q^1,q^0) = g(p_i^1/p_i^0, q_i^1/q_i^0, w_i^0 \mid i = 1,\ldots,n).$

In this expression all the arguments indicated are needed, n+n+(n-1)=3n-1 of them being independent of one another. From (45), a price index formula must not depend on anything but the price relatives, the quantity relatives and the old value shares, and of course, on magnitudes that can be computed with the aid of them. The following magnitudes, for example, may be determined by means of the arguments of g:

(46) the value ratios $\quad v_i^1/v_i^0 = (p_i^1/p_i^0)(q_i^1/q_i^0)$

(47) the ratios $v_i^1/V^0 = (v_i^1/v_i^0)w_i^0$

(48) the total value ratio $V^1/V^0 = \Sigma(v_i^1/V^0)$

(49) the new value shares $w_i^1 = w_i^0(V^0/V^1)(v_i^1/v_i^0)$.

It is shown by straight calculation that the price relatives p_i^1/p_i^0, value shares w_i^1 and w_i^0 and the total value ratio V^1/V^0 determine the arguments of g in (45) and vice versa. Therefore

(50) $\forall f \in \mathcal{F}$: $\exists \varphi$: φ is a function from $\mathbb{R}_+^{3n-1}$ to $\mathbb{R}_+$:

$$f(p^1,p^0,q^1,q^0) = \varphi(p_i^1/p_i^0,\ w_i^1,w_i^0,\ V^1/V^0 \mid i = 1,\dots,n).$$

By interchanging prices and quantities any quantity index formula has the representation

(51) $$f(q^1,q^0,p^1,p^0) = \varphi(q_i^1/q_i^0,\ w_i^1,\ w_i^0,\ V^1/V^0 \mid i = 1,\dots,n).$$

The function φ satisfies the analogs of the commodity reversal test A, the proportionality test D and the identity test E, the formulation of which is straightforward. We will call φ an index number formula as well.

In consequence, any price and quantity index formulas may be expressed in terms of the arguments of φ indicated in (50) and (51). It should be noted that the magnitudes given in them are dimensionless numbers independent of the monetary unit used in valuing the commodities a_i or the units of measurement used to express their amounts.

Thus far, only the general relativity or covariance principle[1] of scientific theories has been applied here, a principle that Albert Einstein formulated for physics as follows: "The laws of physics should be expressed by means of equations that are invariant under any transformation of the space-time coordinates".

For example, Paasche's price index depends exclusively on the magnitudes involved in (50):

$$\frac{p^1 \cdot q^1}{p^0 \cdot q^1} = 1/\Sigma w_i^1 (p_i^1/p_i^0)^{-1} \quad . \tag{52}$$

This is also how Paasche's index is usually computed.

The desiderata 1-6 put forward above relate to the properties of index numbers in pair comparisons $t_0 \rightarrow t_1$ and in combinations of pair comparisons, in which only one and the same, given set of commodities $A = \{a_1, a_2, \ldots, a_n\}$ is dealt with throughout. The desideratum that the index numbers be consistent in aggregation is one that has more rarely been considered, and it has to do with the partition[2] $A_1, \ldots, A_K$ of the commodity

1) Laurikainen (1968) pp. 58-66.

2) Sets $A_1, \ldots, A_K$ make up a partition of the set A if and only if $A = \bigcup_{k=1}^{K} A_k$ and $i \neq j \Rightarrow A_i \cap A_j = \emptyset$.

set A and the properties of the index numbers computed for the various subsets A_k. In order for us to be able to formulate this desideratum, which is mathematically somewhat more complicated than those considered above, somewhat greater notational precision is necessary.

Initially, however, the desideratum concerned will be described.

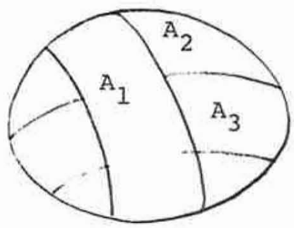
One way of computing the value of, say, a quantity index for a given commodity set A is to do this in two stages. First, the subgroup quantity indices may be evaluated for each subset A_k of the partition $A_1, \ldots A_K$ of the commodity set using the index number formula f chosen. Following this, a total quantity index will be computed from these subgroup indices, by employing the same index number formula.

An index formula is consistent in aggregation if the value of the index as computed via such intermediate stages necessarily coincides with the value obtained by applying the same index number formula directly to the total commodity set A.

The practical idea behind the consistency in aggregation is to simplify the computation of the total quantity index and to provide at the same time a set of subindices which determine the total index. The concept of 'consistent aggregation' used by Theil (1967) p. 159 differs from our concept in that Theil's 'partial index' for A_k is not restricted to be a function of the subset data only but depends in a complicated manner on the <u>total</u> value shares $w_i = v_i/V$ and not only on v_i/V_k.

It seems to me that Theil's 'consistent aggregation' is not a property of an index number formula but a method of aggregation. Should we be smart enough, any formula whatever could be aggregated consistently in this way. We return to these questions in chapter 6.

Figure 1. A partition $A_1,\dots,A_K$ of the set A



In order to avoid confusion with some former concepts of consistency in aggregation, this desideratum will be given an exact formulation, which extensively uses the representations (50)-(51).

Denote the value of the quantity index defined for the total set as follows

$$(53) \qquad Q_0^1 = \varphi(q_i^1/q_i^0,\ v_i^1/V^1, v_i^0/V^0,\ V^1/V^0 \,|\, \{a_i\} \subset A)$$

For each subset A_k of the partition $A_1,\dots,A_k$ the quantity indiced will be defined correspondingly:

$$(54) \qquad Q_0^1(k) = \varphi(q_i^1/q_i^0, v_i^1/V_k^1, v_i^0/V_k^0,\ V_k^1/V_k^0 \,|\, \{a_i\} \subset A_k)$$

These formulas are formally identical with (53). Only the set A has been replaced by the subset A_k and, as a consequence, the total values in the set A have to be replaced by the corresponding "conditional" total values in the set A_k:

(55) $$V = \sum_{a_i \in A} v_i \rightarrow V_k = \sum_{a_i \in A_k} v_i.$$

This of course changes the value shares and value ratios into the corresponding ratios as determined in the set A_k.

After this the subsets A_k will be considered as if they were individual commodities, and a quantity index will be computed with the aid of them for the set A:

(56) $$\bar{Q}_0^1 = \varphi(Q_0^1(k),\ V_k^1/V^1,\ V_k^0/V^0,\ V^1/V^0 \,|\, A_r \subset A).$$

Using this notation, it is possible to formulate the desideratum exactly:

<u>Desideratum 7.</u> Consistency in aggregation:
Let $A_1,\ldots,A_K$ be any partition of the commodity set A.

Then we ought to have $Q_0^1 = \bar{Q}_0^1$ for all $(q^1,q^0,p^1,p^0) \in \mathbb{R}_+^{4n}$.

Consistency in aggregation is formulated analogically for a price index formula, so that it will be a property of $f \in F$ or of the corresponding φ. Consistency in aggregation is a desirable property of the price and volume indices to be computed for, e.g., various commodity groups in national accounting. Provided that the index formula employed is consistent in aggregation, the index for a given total group may be computed directly from the indices for its subgroups, and, conversely, the index for any total group is interpretable as one computed starting from the original commodity level.

It should be pointed out that both Laspeyres's and Paasche's indices are consistent in aggregation, whereas Fisher's ideal index is not exactly so. The value series of national accounting valued at fixed prices form a most simple system consistent in aggregation. The explanation of this consistency lies in the fact that these series may be computed employing either Laspeyres's volume index of Paasche's price index. The widespread use of these two indices is due largely to their aggregation properties, although this has not often been realized. Other indices of the aggregative type, such as, e.g., $f(p^1,p^0,q^1,q^0) = p^1\cdot(q^1+q^0)/p^0\cdot(q^1+q^0)$, are not usually consistent in aggregation.

Of the Vartia Indices I and II to be derived in chapter 6, the first is consistent in aggregation whereas the second is not. Vartia Index I and Stuvel's (1957) index are the only known indices which are consistent in aggregation and satisfy, in addition, both the time and the factor reversal tests.

It should finally be mentioned that, when use is made of an index number formula consistent in aggregation, it will be possible to systematize the price and volume calculations of, say, national accounting in a simple way. As appears from (54)-(56) the computer programs may be made in such a way that, in proceeding toward increasingly aggregative commodity levels, the same index computation program can always be employed, using only the price, quantity and value ratios calculated at the preceding stage.

In Finland, for instance, the methods of computation employed in national accounting for the present are "mixed methods" of a kind, in which different index number formulas are used for different commodity groups and in which they are combined into total indices by means of constant-price value series. Regarding the reliability and interpretation of the results these methods are satisfactory at best.

4. VARIOUS STRATEGIES FOR CONSTRUCTING INDEX SERIES

4.1. Problems of the index series construction

The reliability of index numbers in influenced by numerous factors. L. Törnqvist (1971) p. 53 treats these under two heads:

1) Choice of the method of calculation
2) The data at disposal and lack of data

Here 1) includes the choice of the formula and choice between the base and chain methods. These two choices are independent of each other and therefore it may be misleading to classify the alternatives as Törnqvist (1971) p. 53 does: "The choice of the method of calculation is necessarily a question of judgement. The alternatives will in such a case in practice be Laspeyres', Paasche's and Fisher's methods and some chain index method as the Divisia-Törnqvist method" (underlining ours).

We have elsewhere, see Y. Vartia (1976c), presented a list of 8 'practical' problems[1]) to be solved when starting to calculate, e.g., a consumer price index. Here we will generalize the problem and consider the calculation of any price index whatever. We present a list of 10 questions to which, at least, answers should

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1) v. Hofsten (1952) gives another list of important problems.

be found. These must form a totality which fits the purpose of the index. First the intended use of the index has to be decided on. This decision involves at least the following problems:

(1) Characterization of commodities: What is the general characterization of the set A of commodities whose prices interest us here?

(2) Reference group of economic agents: What will be the group of economic agents (e.g. consumers, producers) from whose point of view the prices are examined?

(3) Length of the time periods: What is the common length of the periods $t_0, t_1, \ldots$ for which the index will be calculated?

We might be interested, say, (1) in the prices of different qualities of paper produced in Sweden and try to measure (2) their average relative change from the point of view of British paper importers (3) quaterly.

Next we list some rather technical problems which have to be solved in order to produce the information necessary for the calculation of the price index. They arise from the fact that the concept of a 'commodity' is not given a priori but must be defined meaningfully according to the situation.

(4) Index commodities: How to classify the commodities in A into disjoint subsets or index commodities $A_1, \ldots, A_k$ in such a way that the quality of each index commodity A_k will stay reasonably stable and the necessary information about it can be estimated?

(5) Price information: How to collect for every period t_m enough price information from the commodities in A, so that the proper price ratios for the A_k:s can be estimated?

(6) Proper weights: How to collect for some period(s) enough information so that proper weights (e.g. means of value shares) for the index commodities can be estimated?

We might use (4) the relevant subgroups of SITC as the operational equivalents A_k of the various qualities of paper and (5) the official data collected by the British foreign trade authorities to calculate the price for A_k as an average unit value.

Here no problems in (6) are present, because the imported quantities of A_k are usually known for every quarter. The price ratios must often be estimated from samples, while the quantities have to be estimated by using expensive market surveys. Price quotations are often only available for some subgroups of A_k:s, so that the estimation of the price ratios for the A_k:s is an index problem in itself.

Now we have attained the situation from which the theory of index numbers usually starts: for every subset A_k there exists a price p_k and a quantity q_k for every period. In chapter 3 we supposed that the complete data is at our disposal. Now we may state shortly the main problem in our study discussed in chapter 3:

(7) Index number formula: How to choose an index number formula in such a way that the information at our disposal will be well utilized?

If we know quantities for A_k only for some periods we cannot use precision formulas of e.g. Fisher or Törnqvist but we have to be satisfied e.g. with Laspeyres' index formula. We have to stress that the various problems we have presented here are strongly interrelated. Some solution for all of them must, however, be found.

But any formula may be applied to calculate price changes over various periods so that the general strategy for constructing the index series is still open:

(8) Strategy for constructing the index series: How to choose the general strategy for constructing the index series from availabl binary comparisons between various periods?

The usual strategies are the 'straight solution' offered by the base method and the chain method. But there are more complicated strategies[1)] which use e.g. increasing symmetrical time intervals $(t_0-h,\ t_0+h)$ centred at some time period t_0 and which utilize in certain cases the information better than, e.g., Laspeyres' index $P_{t_0}^{t_0+h}$ does.

But there are often special problems caused by e.g. quality changes and new or disappearing commodities:

(9) Quality changes: How to take into account the quality changes in our index commodities A_k?

(10) New and disappearing commodities: How to handle new or disappearing commodities?

Here we have interpreted the question of quality changes as a technical problem related to the partition $A_1, \ldots, A_k$ of our A. In some partition of A we might have serious quality changes, which would disappear to a great extent if finer partitions were used. For instance, in our imports data in chapter 7 we have used subgroups of the Standard International Trade Classification (SITC) which in the case of solid fuels contain quite heterogeneous subgroups e.g. coke. These interesting but difficult problems will not, however, be discussed here. Instead, we shall make some short comments about the two rival solutions of (8), namely, the base and chain methods and suppose as before that our data is complete.

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1) I wish to thank L. Törnqvist for pointing out this. See also Allen(1975) p. 145-176.

4.2. Base and chain methods

The ordinary strategy for calculating the average price change from the base period t_0 to the comparison period t_1 is to compare them directly with each other. This is the straight strategy of calculating base indices P_0^1 and Q_0^1. This method is sometimes used even when periods t_0 and t_1 are far from each other. For instance in Finland still in 1976 production price indices[1)] are calculated by comparing monthly price quotations with the mean prices of 1949. The production price index for exported goods is calculated as a weighted average of price ratios the weights being proportional to the fob values of exports in 1949. It is evident that Laspeyres' index used in this way cannot give accurate results, because it completely ignores the changes in the quantities of exported goods. Only if the quantities q_i^t in 1976 are almost proportional to those in 1949 will it work well. The difficulties would become quite evident if we wished to compare modern times with ancient ones, by calculating the change in prices over, say, 3000 years. We would have very few commodities common to both periods. Only chaining would make some sense here[2)], compare e.g. Samuelson and Swamy (1974) p. 587 and Törnqvist (1974) p. 34 and 68. The same applies where the consumer's preferences or the environment is rapidly changing as e.g. in times of war.

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1) See Sahavirta (1970).
Their exact meaning is difficult to comprehend.
No simple answers to, e.g., points (1) and (2) are available.

2) Keynes' (1930, p. 109) opinion is different: "If we want to compile a consumption Index-Number for the value of gold or silver money over the past 3000 years, I doubt if we can do better than to base our composite on the price of wheat and on the price of a day's labour throughout that period. We cannot hope to find a ratio of equivalent substitution for gladiators against cinemas, or for the conveniences of being able to buy motor-cars against the conveniences of being able to buy slaves."

Fisher's (1922) arguments in favour of the base method as against the chain method p. 312 are shortly:

> "On the whole, therefore, the fixed base system (at least as applied to formula 353) is slightly to be preferred to the chain, because,
> (1) it is simpler to conceive and to calculate, and means something clear and definite to everybody;
> (2) it has no cumulative error as does the chain system (as is shown by comparison with Formula 7053);
> (3) graphically it is indistinguishable from the chain system.

His chief argument (2) against the chain method, it seems to me, is a mistaken one. It is true that the chain method gives results different to Fisher's formula 7053, which is a combination (average) of Fisher's ideal indices computed in respect to six base years, see Fisher (1922) p. 301. We give Fisher's figures for 7053 and 353 or Fisher's ideal index calculated both by the base method and by the chain method:

year t	1913	1914	1915	1916	1917	1918
7053	100.00	100.09	99.96	114.03	161.53	177.90
353, Base	100.00	100.12	99.89	114.21	161.56	177.65
353, Chain	100.00	100.12	100.23	114.32	162.23	178.49

But we cannot infer from this that the chain figures are in error! It is perhaps formula 7053 that has a small cumulative error, compare Frisch (1936) p. 9.

Fisher (1922) p. 308 lists (after asserting that "the chain system is of little or no real use") the chief arguments in favour of the chain method:

(1) That it affords more exact comparisons than the fixed base system between the current year and the years immediately preceding in which we are presumably more interested than in ancient history;

(2) that, graphically, the year-to-year lines of the price curve have the correct current directions, whereas in the fixed base system the year-to-year lines are slightly misleading, merely connecting points each of which is really located relatively to the base or origin only, and not to its neighbors; and

(3) that it makes less complicated the necessary withdrawal, or entry, or substitution of commodities, as time and change constantly require.

He admits that it gives the correct comparisons for all consecutive years but, in his opinion, when these are chained together the comparison is not the best one any longer. The best comparison between every two years is the direct comparison between them in Fisher's opinion, see Fisher (1922) p. 299. This would be too simple and most of the modern index theorists do not share Fisher's opinion. The question whether preference should be given to the base or to the chain methods cannot be answered in any simple way, and at an abstract level this problem is in the backround in many notable articles, see e.g. Solow (1957), Richter (1966), Jorgenson and Griliches (1967), Christensen and Jorgenson (1970), Merrilees (1971), Hulten (1973) and Usher (1974). However the variation in cha: figures calculated from good formulas is usually smaller than that in the base figures of the same formulas.

Consider next the chain indices. For the sake of concreteness we will use our GDP price indices of chapter 7 as an example. Any chain index $\bar{P}^t_{1964}$ may be calculated by first calculating year-on-year indices P^t_{t-1} and chaining these according to the formula

(6) $$\bar{P}^{t}_{1964} = \prod_{k=1965}^{t} P^{k}_{k-1} \quad \text{for } t = 1965, 1966, \ldots$$

(7) $$\bar{P}^{t}_{1964} = 1 \quad \text{for } t = 1964$$

(8) $$\bar{P}^{t}_{1964} = 1/\prod_{k=1963}^{t} P^{k}_{k-1} \quad \text{for } t = 1963, 1962, \ldots$$

Another way of expressing the same thing is to define $\bar{P}^{1964}_{1964} = 1$ and calculate $\bar{P}^{t}_{1964}$:s recursively, using the equations

(9) $$\bar{P}^{t+1}_{1964} = (\bar{P}^{t}_{1964}) \cdot (P^{t+1}_{t}) \quad \text{, forward step: } t \geq 1964$$

(10) $$\bar{P}^{t-1}_{1964} = (\bar{P}^{t}_{1964})/(P^{t}_{t-1}) \quad \text{, backward step: } t \leq 1964$$

However the most elegant definition - and a general one - of a chain index P^{t}_{s} from year s to year t is as follows:

(11) $$\log \bar{P}^{t}_{s} = \log P^{s+1}_{s} + \log P^{s+2}_{s+1} + \ldots + \log P^{t}_{t-1}, \quad s<t.$$

(12) $$\log \bar{P}^{t}_{s} = - \log \bar{P}^{s}_{t} \quad , s>t$$

To get the log-change of prices from year s to year t (as calculated by the chain index) just add together the consecutive yearly log-changes of prices.[1)] This construction guarantees that every chain index has the circular property $\bar{P}^{t}_{s} = \bar{P}^{k}_{s}\bar{P}^{t}_{k}$ for all s, k and t and the time reversal property $\bar{P}^{t}_{s} = 1/\bar{P}^{s}_{t}$ for all s and t, irrespective of the choice of the d.c. index number formula f used in P^{t}_{t-1}:s. But there is nothing to guarantee that the weak identity property,

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1) This is essentially the same as Frisch's (1936) definition. See also Banerjee (1975) p. 55.

i.e. $p^t = p^s$ and $q^t = kq^s$ implies $\bar{P}_s^t = 1$, or the weak proportionality property, i.e. $p^t = kp^s$ and $q^t = q^s$ implies $\bar{P}_s^t = k$, is satisfied. The same applies to the Divisia - Törnqvist's integral formula, to be considered in the next chapter, as e.g. Törnqvist (1974) p. 30 has noted.

The formula (11) corresponds to time intervals (s,s+1), (s+1, s+2),...,(t-1, t) which may be, e.g., consecutive years as they are in our chain index calculations in chapter 7. But we may in principle divide the years into quarters, quarters into months, months into days, etc. and calculate a chain index for every partition of our time period. In practise calculations of this kind become soon impossible because of lack of data for short periods. But let us imagine that we have hypothetical values $v_i(t)$ and quantities $q_i(t)$ which are continuous functions of time $t \in (a,b)$; i.e., for every commodity a_i we have two continuous functions v_i and q_i which map the time interval (a,b) to nonnegative real numbers. Put $p_i(t) = v_i(t)/q_i(t)$ and suppose that $q_i(t) = 0$ only when $v_i(t) = 0$ so that $p_i(t)$ is defined for all points where $q_i(t)>0$. For any time interval $(t-\tau/2,\ t+\tau/2) \subset (a,b)$ define

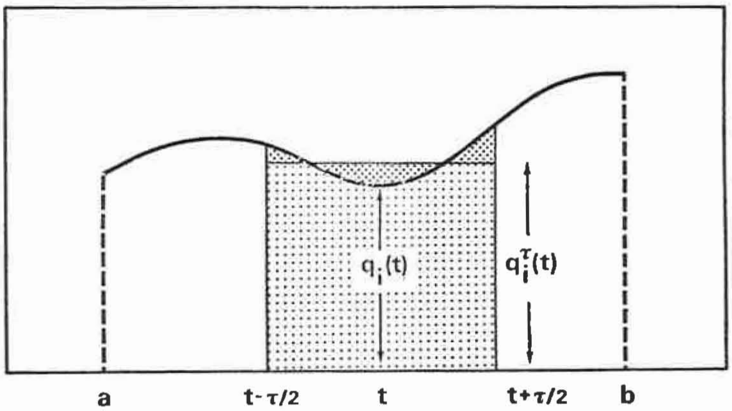
$$v_i^\tau(t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} v_i(t)\,dt \tag{13}$$

$$q_i^\tau(t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} q_i(t)\,dt \tag{14}$$

$$p_i^\tau(t) = v_i^\tau(t)/q_i^\tau(t) \quad . \tag{15}$$

Here $v_i^\tau(t)$, $q_i^\tau(t)$ and $p_i^\tau(t)$ are the average value, quantity and price of commodity a_i for period of length τ and centred at t.

Figure 1: Definition of $q_i^\tau(t)$.



Let $t_1, t_2, \ldots, t_k$ be the meanpoints of K disjoint time intervals of length τ, whose union is (a,b). Define e.g.

$$(16) \qquad P_k^{k+1} = \Sigma w_i^\tau(t_k)\,[p_i^\tau(t_{k+1})/p_i^\tau(t_k)], \text{ where}$$

$$w_i^\tau(t_k) = v_i^\tau(t_k)/\Sigma_j v_j^\tau(t_k) \quad .$$

These are Laspeyres' indices calculated from the data (13)-(15) for consecutive time intervals. Define according to (11)

$$(17) \qquad \log \bar{P}_1^K = \sum_{k=1}^{K-1} \log P_k^{k+1} \quad .$$

When $K \to \infty$ log $\bar{P}_1^K$ approaches (under some reqularity conditions) the value

(18) $$\sum_i \int_a^b w_i(t) d \log p_i(t) \quad , \text{ where}$$

(19) $$w_i(t) = v_i(t) / \sum_j v_j(t) \quad .$$

This is the integral form of Divisia-Törnqvist's index. Many other index formulas in addition to Laspeyres' formula in (16) give the same limit (18). Thus if there existed continuous functions $v_i(t)$, $q_i(t)$ and $p_i(t)$ and if the observed values, quantities and prices were their means as defined by (13)-(15) then many chain indices log $\bar{P}_1^K$ defined by (17) would approximate the theoretical Divisia-Törnqvist's index log $\frac{P(t_K)}{P(t_1)}$ defined by (18) for $a = t_1$ and $b = t_K$. But because of the discontinuous character of most economic phenomena (at least transactions) continuous values, quantities and prices do not actually exist, unless we intentionally construct them as some kind of smoothed series (e.g. moving averages) as we will do in the next chapter. This, however, complicates the situation considerably as the value of (18) will depend on the smoothing procedure.

5. DIVISIA - TÖRNQVIST'S INDEX

5.1. General

The considerations in the previous chapters dealt with period analysis, where time is divided to periods of considerable length. Prices, quantities and values are regarded as constants within each period. Another theory of index numbers developed independently by Francois Divisia (1925) and Leo Törnqvist (1936), starts from the idea that prices, quantities and values are defined for every point of time and are more or less continuosly changing functions of time. Only descriptive or statistical aspects of Divisia-Törnqvist's theory will be dealt with here.

The considerations in chapter 3 may be interpreted as relating to the sales of a commodity exchange in two periods t_0 and t_1 (either imagined or real) of a year's duration. For each of the two years there is a commodity basket consisting of the same commodities, and these two baskets are compared. In the present chapter the analysis will be made more comprehensive in the time dimension, and an effort will be made to define the concepts of price, quantity and the value of sales for each point of time.

5.2. Defining values, quantities and prices for all points of time

Let us consider two points of time, t and t+τ, expressed in terms of years, and let $\bar{v}(t,t+\tau)$ be the value of the sales of the commodity a, in terms of marks, in the time interval concerned. Expressed in marks per year the sales during this time interval will be $\bar{v}(t,t+\tau)/\tau$, which is the average annual rate of sales in the time interval $(t,t+\tau)$.

As the length τ of the time interval tends to zero, we get for the instantaneous annual rate of sales at the point of time t:

$$(1) \qquad v(t) = \lim_{\tau\to 0}\bar{v}(t,t+\tau)/\tau \ .$$

This is the ordinary time derivative of the sales provided that the sales $\bar{v}(t)$ are measured as the total sales since a given point of time (e.g., the point of time when the commodity exchange was founded):

$$(2) \qquad v(t) = \frac{d}{dt}\,\bar{v}(t) = \lim_{\tau\to 0}(\bar{v}(t+\tau)-\bar{v}(t))/\tau \ .$$

The latter expression corresponds to the procedure followed, e.g., in kinematics, where the system of space co-ordinates may be chosen arbitrarily. In the case of economic phenomena the situation is rendered troublesome by the fact that, if a point of time other than the one at which the commodity exchange was established is chosen as the "origin", sales will be formally negative at all points of time earlier than this.

Correspondingly, the quantities of the particular commodity a sold at the commodity exchange during the time interval $(t,t+\tau)$ will be denoted by $\bar{q}(t,t+\tau)$, and thus the annual rate of sales at the point of time t, expressed in physical units, will be

$$(3) \qquad q(t) = \lim_{\tau \to 0} \bar{q}(t,t+\tau)/\tau \ .$$

The average unit prices in marks will be $\bar{p}(t,t+\tau) = \bar{v}(t,t+\tau)/\bar{q}(t,t+\tau)$, and these will be rather independent of the length τ of the time interval.

Correspondingly, the instantaneous price at the point of time t will be

$$(4) \qquad p(t) = \lim_{\tau \to 0} \left(\frac{\bar{v}(t,t+\tau)}{\bar{q}(t,t+\tau)}\right) = \frac{v(t)}{q(t)} \ .$$

The latter equation is obtained by dividing both the numerator and denominator in the limit operation by τ.

An assumption underlying (1)-(4) is that, e.g., the function $\bar{v}(t,t+\tau)$ in (1) can be "smoothed" at its points of discontinuity: the increment in sales can be distributed evenly between, say, time intervals of a second's length. A more sophisticated application of this idea leads to the use of so-called Dirac delta functions[1)] or to other corresponding

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1) See, e.g., Zadeh & Desoer (1963).

mathematical methods such as the distribution calculus[1] or the non-standard analysis[2].

What guarantees that the mean values $\bar{v}(t,t+\tau)/\tau$, $\bar{q}(t,t+\tau)/\tau$ and $\bar{p}(t,t+\tau)$ will approach some well-defined and empirically meaningful, say, continuous functions when the length of the time period τ diminishes? Nothing: they do not usually approach any continuous functions. These average values become, however, smoother and more continuous as τ increases.

If both the instantaneous value and volume of sales at any one point of time are known, the corresponding accumulated values are obtained by integration as follows:

$$\text{(5)} \qquad \bar{v}(t,t+\tau) = \int_t^{t+\tau} v(t)dt$$

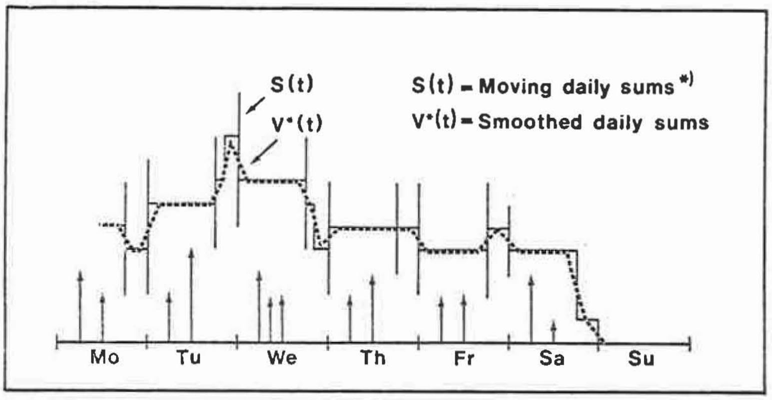
$$\text{(6)} \qquad \bar{q}(t,t+\tau) = \int_t^{t+\tau} q(t)dt \ ,$$

the average price being obtained as a ratio of these two.

As far as the instantaneous value v(t) is determined from (1), its graph for a week's interval may be as follows:

1) See, e.g., Zemanian (1965).
2) See Robinson (1966).

Figure 1: Smoothing of the instantaneous value v(t)



*) The vertical line segments in the graph of S(t) represent two alternative movement possibilities

The arrows represent Dirac delta functions as multiplied by the increment in the sales at the point of time t. The dotted curve in the figure represents the smoothed daily sums

$$(7) \qquad V^*(t) = \frac{1}{h}\int_{t-h/2}^{t+h/2} \bar{v}(t-\tau/2,t+\tau/2)\,dt \;, \quad h<\tau$$

$$= \frac{1}{h}\int_{t-h/2}^{t+h/2} S(t)\,dt$$

determined with the aid of the moving daily sums

$$(8) \qquad S(t) = \int_{t-\tau/2}^{t+\tau/2} v(t)\,dt = \bar{v}(t-\tau/2,t+\tau/2) \;, \text{ where } \tau \approx \frac{1}{365} \;.$$

The smoothed value $V^*(t)$ does not react as easily to daily or other short-term variations as does the unadjusted rate. (In determining the speed of an airplane, neither perturbations nor the vibrations taking place in accordance with the sound waves are taken into account!). The moving daily sum S(t) is a step function but the smoothed value $V^*(t)$ is a first order spline function.

5.3. Derivation of Divisia-Törnqvist's indices

Consider the sales of the set of commodities $A = \{a_1, a_2 \ldots, a_n\}$ over the interval $(t, t+\tau)$:

$$\text{(9)} \qquad \bar{V}(t,t+\tau) = \Sigma\, \bar{v}_i(t,t+\tau).$$

Dividing by the length of the interval, we get the average value of sales:

$$\text{(10)} \qquad \bar{V}(t,t+\tau)/\tau = \Sigma\, \bar{v}_i(t,t+\tau)/\tau,$$

which may be denoted

$$\text{(11)} \qquad V(t,t+\tau) = \Sigma\, v_i(t,t+\tau) = \Sigma\, p_i(t,t+\tau)q_i(t,t+\tau).$$

Taking natural logarithms and differentiating we have (cancelling dt):

$$\text{(12)} \qquad \frac{dV(t,t+\tau)}{V(t,t+\tau)}$$

$$= \Sigma \frac{[dp_i(t,t+\tau)]q_i(t,t+\tau)+p_i(t,t+\tau)[dq_i(t,t+\tau)]}{\Sigma p_i(t,t+\tau)q_i(t,t+\tau)}$$

$$= \Sigma \frac{v_i(t,t+\tau)}{V(t,t+\tau)} \frac{dp_i(t,t+\tau)}{p_i(t,t+\tau)} + \Sigma \frac{v_i(t,t+\tau)}{V(t,t+\tau)} \frac{dq_i(t,t+\tau)}{q_i(t,t+\tau)} .$$

This may be rewritten as

$$\text{(13)} \qquad d\log V(t,t+\tau) = \Sigma\, w_i(t,t+\tau) d\log p_i(t,t+\tau) +$$

$$\Sigma\, w_i(t,t+\tau) d\log q_i(t,t+\tau) .$$

Thus, when the infinitesimal changes in prices and volumes are weighted by the average value shares for the period $(t,t+\tau)$, the infinitesimal relative change in sales is obtained. The equation is suitable for the analysis of the changes in prices and quantities, as well as for the analysis of the consequences of alternative price and quantity situations.

A corresponding equation also holds good for the instantaneous prices, quantities and values when certain conventions are made:

$$(14) \qquad d\log V(t) = \Sigma\, w_i(t) d\log p_i(t) + \Sigma\, w_i(t) d\log q_i(t) \ .$$

These equations are obtained from (13) by letting τ approach zero.

Substituting the expression of V(t) in terms of hypothetical price and quantity factors , cf Törnqvist (1935,1937,1974),

$$(15) \qquad V(t) = P(t)Q(t),$$

we have

$$(16) \qquad d\log P(t) + d\log Q(t) = \Sigma\, w_i(t) d\log p_i(t) + \Sigma\, w_i(t) d\log q_i(t) \ .$$

Here it is natural to define the infinitesimal relative changes in the price and volume indices by

(17) $$d\log P(t) = \Sigma\, w_i(t)\, d\log p_i(t)$$ and

(18) $$d\log Q(t) = \Sigma\, w_i(t)\, d\log q_i(t)\ .$$

These are the Divisia-Törnqvist indices in differential form. Thus the infinitesimal relative change in a price (or, alternatively, a volume) index is the weighted average of the corresponding price (volume) changes, the weights being the value shares of the commodities concerned at the point of time under consideration.

Also, for the average prices in the time interval $(t,t+\tau)$, for instance, a similar equation is obtained:

(19) $$d\log P(t,t+\tau) = \Sigma\, w_i(t,t+\tau)\, d\log p_i(t,t+\tau),$$ where

(20) $$w_i(t,t+\tau) = v_i(t,t+\tau)/V(t,t+\tau).$$

Thus, the system of weights consists here of the average value shares of the commodities a_i during the period under consideration.

Equations (17) and (18) indicate the effects on the price and volume indices of infinitesimal changes in prices and quantities related to a given point of time. By integration we obtain the log-changes over the time interval (a,b) in the price and volume indices that depend on the price and quantity movements, considered as a whole:

(21) $$\log\frac{P(b)}{P(a)} = \Sigma \int_a^b w_i(t)d\log p_i(t)$$

(22) $$\log\frac{Q(b)}{Q(a)} = \Sigma \int_a^b w_i(t)d\log q_i(t) \ .$$

These are the Divisia-Törnqvist indices in integral form.

In these formulas the symbols a and b have been used on purposes to denote points of time, in contradistinction to the symbols t_0 and t_1 employed before to denote periods of time.

Formula (22), for example, indicates how the log-change in quantity between the points of time a and b - i.e., the log-change from Q(a) to Q(b) - can be obtained from the changes in the volume and value shares of the individual commodities a_i. This log-change will tend to minus infinity as the volume at the point of time b, Q(b), tends to zero. Provided that the price factor $P(b) \geq c > 0$, the volume will tend to zero together with $V(b) = P(b)Q(b)$. If the volume index is computed from the instantaneous volumes $q_i(t)$, it will be almost invariably equal to zero, and it will have little to do with ordinary index number calculations.

The price of a commodity a_i can be defined for every point of time, whereas the value and volume of its sales will, by the customary definition, be zero at almost every point of time. This difficulty may be avoided by determining the value and volume as, e.g., an average weekly time rate[1]. Weekly averages are usually determined only for calendar weeks, numbering about 52 in a year. However, nothing prevents us from computing weekly averages for each of the 7-day periods which, in an ordinary 365-day year, are 359 in number. In principle, weekly averages may be determined however frequently, e.g., for every period of a week's length beginning at a full hour of any of the 365 days of a year, but in practice the lack of hourly data often makes such calculations impossible.

Denote the average value and volume of the sales of the commodity a_i during the period $(t-\tau/2, t+\tau/2)$ centred at t and of length τ respectively by

(23) $$v_i^\tau(t) = v_i(t-\tau/2, t+\tau/2)$$

(24) $$q_i^\tau(t) = q_i(t-\tau/2, t+\tau/2) \; .$$

Provided that the sales are referred to in their entirety to the point of time of closing the deal, the average value and volume will be step functions continuing from the right and, thus, discontinuous. Dividing the value and volume evenly between intervals of, say, a second's length, however, the

1) Should months instead of weeks be considered, the variable length of the calendar months would give rise to a furhter difficulty.

functions (23) and (24) can be rendered continuous and differentiable[1)], at the same time that their empirical interpretation is left intact. The average unit price of the commodity a_i for the time interval $(t-\tau/2, t+\tau/2)$ is

$$(25) \qquad p_i^\tau(t) = v_i^\tau(t)/q_i^\tau(t).$$

In the following we shall assume that the functions (23)-(25) are continuous and differentiable[1)] functions of time. These make it possible to rewrite the equations (21) and (22) in a more accurate form as

$$(26) \qquad \log\frac{P^\tau(b)}{P^\tau(a)} = \Sigma \int_a^b w_i^\tau(t) d\log p_i^\tau(t)$$

$$(27) \qquad \log\frac{Q^\tau(b)}{Q^\tau(a)} = \Sigma \int_a^b w_i^\tau(t) d\log q_i^\tau(t) \quad .$$

The value index can also be expressed in the same form:

$$(28) \qquad \log\frac{V^\tau(b)}{V^\tau(a)} = \Sigma \int_a^b w_i^\tau(t) d\log v_i^\tau(t) \quad ,$$

and thus the factor reversal test will be met at any positive choice of τ. The index numbers thus defined will depend largely, like the magnitudes (23)-(25), on the choice of the length parameter τ of the time period. However, most of this

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1) More precisely, differentiable almost everywhere, see figure 1 on p. 107.

dependence is of a trivial kind, because for $V^{\tau}(t)$, for example, we have

$$(29) \qquad V^{\tau}(t) \approx \left(\frac{\tau_1}{\tau}\right)\int_{t-\tau/2}^{t+\tau/2} V^{\tau_1}(t)dt,$$

provided that $\tau >> \tau_1$. This is to say that the "long-term average" of $V^{\tau}(t)$ can be estimated as the time average of the "short-term averages" $V^{\tau_1}(t)$. The same applies to the volume term $Q^{\tau}(t)$, and the price term $P^{\tau}(t)$ can be estimated as the ratio between these two.

Subsequently, however, the parameter τ will be omitted, assuming at the same time that it will be kept unchanged and sufficiently large. The effects of the length τ of short time periods on index number formulas would be particularly called for.

5.4. Properties of Divisia-Törnqvist's indices

The problem of so-called vanishing commodities is one that may be analysed by means of the integral forms of Divisia-Törnqvist index numbers. Let us assume that the commodity a_i disappears from the market in such a way that $p_i(b) \to 0$, but $q_i(b) \to q_i > 0$, whereas the prices and quantities of all the other commodities a_i remain constant and equal p_j and q_j respectively. The effect of an extreme decline in the price of a_i on the change of the price index will be

$$(30) \qquad \int_a^b w_i(t)d\log p_i(t) = \int_a^b \frac{q_i(t)dp_i(t)}{\Sigma p_j(t)q_j(t)}$$

$$\approx \frac{q_i}{\Sigma p_j q_j} \int_a^b p_i(t) \to -\frac{p_i(a)q_i}{\Sigma p_j q_j}, \text{ when } p_i(b) \to 0.$$

Thus the effect of an extreme decline in the price of a_i on the log-change in the price index is approximately equal to the negative of its value share $w_i(a)$ at the point of time a. The index formulae to be presented later can be tested with the aid of this result.

To make the exact meaning of the above integrals evident and to render broader interpretations possible it is useful to rewrite equations (26)-(28) using another notation. The notation to be used below is that of Apostol (1957) p. 277. Let

(31) $\alpha_r(t) = \log p_r(t)$, $r = 1,2,\ldots,n$

(32) $f_r(\vec{\alpha}(t)) = w_r(t)$

$= \exp[\log p_r(t)+\log q_r(t)-\log\Sigma p_r(t)q_r(t)]$

(33) $\Gamma(\vec{\alpha})$ is a curve in $\mathbb{R}^n$ defined by the vector $\vec{\alpha}(t)$, where $t \in (a,b)$.

Employing this notation, (1) can be rewritten in the form

$$(34) \qquad \log\frac{P(b)}{P(a)} = \sum_{r=1}^{n} \int_a^b f_r(\vec{\alpha}(t))d\alpha_r(t),$$

where all the terms are Stieltjes integrals. Following Apostol, this can further be written as a line integral

$$(35) \qquad \log\frac{P(b)}{P(a)} = \int_{\Gamma(\vec{\alpha})} \vec{f}\cdot d\vec{\alpha} \ ,$$

provided only that the vector function $\vec{\alpha}(t) = (\alpha_1(t),\ldots,\alpha_r(t))$ is continuous, and this was ensured by the interpretations introduced above. By (35) the log-change in the prices is the line integral of the value shares $w_r(t)$ computed along the curve defined by the logarithms of the prices, $\log p_r(t)$. Essential is that not only the end points of the curve but also its course may affect the result.

In evaluating the integrals in (34) use may be made of the mean value theorem of the Riemann-Stieltjes integrals, see Apostol (1967) p. 213:

$$(36) \qquad \int_a^b f_r(\vec{\alpha}(t))d\alpha_r(t) = \int_a^b w_r(t)d\log p_r(t)$$

$$= \bar{w}_r\int_a^b d\log p_r(t)$$

$$= \bar{w}_r\log\frac{p_r(b)}{p_r(a)} \ .$$

Here we suppose that the prices either increase or decrease over the interval [a,b]. As the value shares $w_r(t)$ are continuous functions of time, $\bar{w}_r$ will be the value share $w_r(\bar{t})$ at some particular point of time $\bar{t}$ in the interval [a,b]. In any case, $\bar{w}_r$ will be the following average value share (weighted by the price changes):

(37) $$\bar{w}_r = \int_a^b w_r(t)\,d\log p_r(t) / \int_a^b d\log p_r(t).$$

Substituting this into (41) we have

(38) $$\log\frac{P(b)}{P(a)} = \Sigma\, \bar{w}_r \log\frac{p_r(b)}{p_r(a)} ,$$

which shows that a solution of a given simple type exists for the Divisia-Törnqvist indices in integral form, provided that all prices change monotonically.

An expression of the corresponding type is obtained for the volume index:

(39) $$\log\frac{Q(b)}{Q(a)} = \Sigma\, \hat{w}_r \log\frac{q_r(b)}{q_r(a)} .$$

The average value shares $\hat{w}_r$ occuring here depart, however, usually from the shares $\bar{w}_r$ because the quantity log-changes in

(40) $$\hat{w}_r = \int_a^b w_r(t)\,d\log q_r(t) / \int_a^b d\log q_r(t).$$

are the weighting functions.

The index number problem can thus be reduced to the problem of the choice of the weights $\bar{w}_r$ and $\hat{w}_r$. This is because, by (38) and (39), the log-changes in price and volume indices may, under certain conditions, be expressed as a linear combination of the log-changes of individual prices or quantities, in which the coefficients are means of the value shares for the time interval under consideration.

It should be noted that the weights $\bar{w}_r$ or $\hat{w}_r$ need not necessarily add up to unity, despite the fact that, for each point of time, $\Sigma\, w_r(t) = 1$. It will later be observed that, in approximating (38) and (39), it is useful to get rid of the ordinarily used interpretation of a mean in which the sum of the weights is normed so as to equal unity.

Divisia - Törnqvist's integral formula leads to price and volume indices in which the weigths usually depart from each other. In consequence, the indices (38) and (39) thus arrived at do not satisfy the "factor reversal" test. If the time interval [a,b] is comparatively long and there is enough information available on the movements of prices and quantities, different weights $\bar{w}_r$ and $\hat{w}_r$ may be used. An explicit use of mutually different weights can, however, be avoided by dividing the interval [a,b] into parts and computing, e.g., the integral (36) by parts. This is, in an abstract form, the chain principle advocated particularly by Törnqvist. Let $a = t^1 < t^2 < \ldots < t^{K+1} = b$. Then it will hold true (provided that the prices change monotonically over each of the sub-intervals)

$$(41)\qquad \int_a^b w_r(t)\,d\log p_r(t) = \sum_{k=1}^{K} \int_{t_k}^{t^{k+1}} w_r(t)\,d\log p_r(t)$$

$$= \sum_{k=1}^{K} w_r(\bar{t}_k)\log\frac{p_r(t^{k+1})}{p_r(t^k)} ,$$

where $t^k \leq \bar{t}^k \leq t^{k+1}$. Now we may further define

$$(42)\qquad \bar{w}_r = \frac{w_r(\bar{t}^1)\log\frac{p_r(t^2)}{p_r(t^1)} + \ldots + w_r(\bar{t}^K)\log\frac{p_r(t^{K+1})}{p_r(t^K)}}{\log\frac{p_r(b)}{p_r(a)}} ,$$

or in other words, $\bar{w}_r$ will be the mean of the value shares in the sub-intervals, $w_r(\bar{t}^k)$ as weighted by the price changes. Equations corresponding to these hold true for the volumes.

If the time interval [a,b] has been divided into sub-intervals so short that price and volume movements within these sub-intervals are not known, it is warranted to use the same weights for both price and quantity changes. When the sub-intervals are short, the error of approximation will be small in any case.

Finally we shall briefly consider the dependence of the log-change of the price index (35) on the path of integration $\Gamma(\vec{\alpha})$. As is well known, the change in a price index need not necessarily be zero even though $\alpha_r(a)=\log p_r(a)=\log p_r(b)=\alpha_r(b)$, i.e. $\Gamma(\vec{\alpha})$ is a Jordan curve.

This peculiar state of affairs can be seen, e.g. from (42), according to which the effect on or the contribution to the log-change in a price index of an individual price change is

$$\bar{w}_r \log\frac{p_r(b)}{p_r(a)} = w_r(\bar{t}^1)\log\frac{p_r(t^2)}{p_r(t^1)} + \ldots + w_r(\bar{t}^K)\log\frac{p_r(t^{K+1})}{p_r(t^K)} . \tag{43}$$

The left-hand side of this equation is defined with the aid of its right-hand side. The right-hand side may take on values different from zero even when $p_r(b) = p_r(a)$. The left-hand side will then be of the form $\pm\infty\cdot 0$, or the weight $\bar{w}_r$ (no longer interpretable as a value share!) will attain an infinitely large value. Thus the commodity a_r may have an effect different from zero on the change in the price index, despite the fact that the total change in its price is zero. Because of this peculiarity, for instance, it is reasonable to divide the time interval [a,b] into shorter intervals for the application of the chain principle.

Under what conditions will the change in the price index (35) be independent of the path of integration? No general results are known if the value shares (32) are any arbitrary nonnegative functions of time the sum of which equals one. They should be functions of the price vector only, i.e., they should not depend independently on the quantity vector or the total income. In this case we have a precise answer to this question given by a theorem of Apostol (1957) p. 280:

Theorem. Assume that the vector of value shares, $\vec{f} = (w_1,\ldots,w_n)$, is a continuous mapping of the logarithmic prices $x_i = \log p_i$ defined in an open region S of $\mathbb{R}^n$ and that there exists a real-valued and differentiable mapping $\emptyset$ defined in S for which

(44) $\forall \vec{x} : \vec{x} \in S \Rightarrow \vec{f}(\vec{x}) = \nabla \phi(\vec{x})$.

Then, for every point $\vec{x}$ and $\vec{y}$ of S and for every price-wise smooth curve Γ joining $\vec{x}$ and $\vec{y}$ we have

(45) $$\int_{\Gamma(\vec{x},\vec{y})} \vec{f}\cdot d\vec{\alpha} = \int_{\Gamma(\vec{x},\vec{y})} \nabla\phi\cdot d\vec{\alpha} = \phi(\vec{y})-\phi(\vec{x}),$$

where $\vec{\alpha} \in \Gamma(\vec{x},\vec{y})$.

The function ϕ is a <u>potential function</u> of the value share vector $\vec{f}$ (which function here corresponds to the logarithm of the price index, logP(t)). It can be shown, conversely, that independence of the path of integration implies the existence of a potential function. Apostol (1957) gives, on p. 293, an interesting necessary condition, concerned with the partial derivates of the value shares, for the existence of the potential function:

(46) $$D_i f_j(\vec{x}) = D_j f_i(\vec{x}) .$$

Written in another notation this becomes

(47) $$\frac{\partial w_j}{\partial \log p_i} = \frac{\partial w_i}{\partial \log p_j} .$$

Further, by using (32)

(48) $$\frac{\partial w_j}{\partial \log p_i} = w_j \left[e_{ji} - \sum_r w_r e_{ri} - w_i\right] ,$$

where $e_{ji} = \partial \log q_j / \partial \log p_i$ is the elasticity of the quantity of demand for a_j with respect to the price p_i. Assume that the prices $p_i(t)$, $i=1,2,\ldots,n$ and the income $V(t)$ are exogenous variables and that the quantities $q_i(t)$, $i=1,2,\ldots,n$, will at every point of time immediately adjust to these exogenous factors, in the way postulated by the classical theory of demand[1]. It is interesting to examine the contention, presented to me, that in this situation the integral (45) will be independent of the path of integration. Equation (47) can now be rewritten with the aid of (48) as $w_i e_{ij} = w_j e_{ji}$, since, according to the classical demand theory (see, e.g., Henderson and Quandt (1971) p. 39), $\Sigma\, w_r e_{ri} = -w_i$. Nevertheless, this does not always hold true, but, according to the classical demand theory, $w_i e_{ij} = w_j e_{ji} + w_i w_j (\eta_j - \eta_i)$, which is the so-called Slutsky equation as written in terms of elasticities and shares (see Malinvaud (1972) p. 37).

Thus the integral (45) is not necessarily independent of the path of integration even under the classical theory of demand, because the value shares are usually dependent on income. A necessary additional condition is that all the income elasticities η_i be equal to one. This corresponds to the homothetic case, where everything is nice and simple as Samuelson and Swamy (1974) show. Afterwards we have found, to our surprise, that Hulten (1973) has proved the path independence of the Divisia-Törnqvist's index by referring to the same theorems of Apostol (1957)! Hulten's interest is, however, in the production theory whereas our's is here in the demand theory.

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1) This expression has not, of course, any completely well-established meaning. Of the results of the classical demand theory, mainly those concerning the properties of the demand functions are needed here, the point of departure for the present analysis being the situation considered in E. Malinvaud (1972) chapter 9.

6. NEW INDEX NUMBER FORMULAS BASED ON RELATIVE CHANGES

6.1. General

Let us return to the problem that was put aside at the end of chapter 1. We want to determine the relative change in a sum with the aid of the relative changes in the terms of the sum. Interest is of course focused particularly on the log-change as an indicator of the relative change. To make the presentation more illustrative[1] we will consider the accounting identity

$$Y^t = C^t + I^t \qquad (1)$$

which says that the value of income Y^t for some period t equals the values of consumption C^t and investment I^t for the same period.

Subtracting equations (1) for two periods (or any two situations) t_1 and t_0 we get

$$Y^{t1} - Y^{t0} = C^{t1} - C^{t0} + I^{t1} - I^{t0}, \text{ or } \Delta Y = \Delta C + \Delta I . \qquad (2)$$

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1) This was how Vartia Index I was actually discovered, in constructing a solution program for the short-term econometric model of Etla, reported by P. Vartia (1974). Here accounting identities were to be written in terms of relative changes, see Y. Vartia (1974).

By dividing and multiplying by any nonzero means $\hat{Y}$, $\hat{C}$, $\hat{I}$ of the values of Y, C and I we get

(3) $$\frac{\Delta Y}{\hat{Y}} = \left(\frac{\hat{C}}{\hat{Y}}\right)\frac{\Delta C}{\hat{C}} + \left(\frac{\hat{I}}{\hat{Y}}\right)\frac{\Delta I}{\hat{I}} .$$

If we take, e.g., the 'mean' $\hat{Y} = Y^{t0}$ and similarly for $\hat{C}$ and $\hat{I}$ we have the familiar identity for an ordinary relative change: the relative change in Y is a weighted average of the relative changes in C and I, the weights being the 'old' value shares C^{t0}/Y^{t0} and I^{t0}/Y^{t0}. Specifying different means $\hat{Y} = K(Y^{t1}, Y^{t0})$, etc. we get similar identities for various indicators $H(y/x) = (y-x)/K(y,x)$ of relative change. Especially using the logarithmic mean $\hat{Y} = L(Y^{t1}, Y^{t0})$, etc. and the representation (27) of chapter 1 we get for the log-change

(4) $$\log \frac{Y^{t1}}{Y^{t0}} = \frac{\hat{C}}{\hat{Y}} \log \frac{C^{t1}}{C^{t0}} + \frac{\hat{I}}{\hat{Y}} \log \frac{I^{t1}}{I^{t0}} .$$

This is the needed decomposition for the logarithmic change of the sum (1).

6.2. The Vartia Index I

Making use of the notations introduced in chapters 1 and 3 we have similar decompositions for a general sum $V = \Sigma v_i = \Sigma p_i q_i$ and any indicator $H(\frac{y}{x}) = \frac{y-x}{K(y,x)}$ of the relative change

(5) $$H\left(\frac{V^1}{V^0}\right) = \sum \frac{K(v_i^1, v_i^0)}{K(V^1, V^0)} H\left(\frac{v_i^1}{v_i^0}\right) .$$

Especially for the log-change

(6) $$H_4\left(\frac{y}{x}\right) = \frac{y-x}{L(y,x)} = \log \left(\frac{y}{x}\right)$$

we have

(7) $$\log(\frac{V^1}{V^0}) = \sum \frac{L(v_i^1, v_i^0)}{L(V^1, V^0)} \log(\frac{v_i^1}{v_i^0})$$

$$= \sum w_i \log(v_i^1/v_i^0)$$

using an obvious notation. Because the log-change of the value $v_i = p_i q_i$ equals the sum of the log-change for price p_i and quantity q_i,

(8) $$\log(v_i^1/v_i^0) = \log(p_i^1/p_i^0) + \log(q_i^1/q_i^0) \quad ,$$

we have identically

(9) $$\log(V^1/V^0) = \log P_0^1 + \log Q_0^1$$

if we define the log-changes of the price and quantity indices in (9) in the following natural way:

(10) $$\log P_0^1 = \sum_{i=1}^{n} w_i \log(p_i^1/p_i^0)$$

(11) $$\log Q_0^1 = \sum_{i=1}^{n} w_i \log(q_i^1/q_i^0) \quad , \text{ where}$$

(12) $$w_i = \frac{L(v_i^1, v_i^0)}{L(V^1, V^0)} = \frac{\hat{v}_i}{\hat{V}} \, .$$

According to (7) a decomposition similar in form to these also holds good for the value $V = \Sigma v_i = \Sigma p_i q_i$. This derivation is the discrete analog of the derivation of Divisia-Törnqvist's indices (17) and (18) in chapter 5. The index number formula defined by (10) or (11) will be called the Vartia Index I.

Equation (9) shows that the our new index formula, Vartia Index I, meets the factor reversal test. It is a routine task to show that the formula (10) defines a price index in the sense of chapter 3 and satisfies the time reversal test. Its other properties will be discussed later.

A feature characteristic of the Vartia Index I thus defined is that the sum of the weights w_i does not usually equal unity. In Appendix 4 it is shown that Σw_i is less than or equal to unity and that it will equal unity only if $w_i^0 = w_i^1$ for every commodity a_i. In this case relative changes in value for all commodities will be equal.

Ordinarily the sum of the weights of the index number formulas of this type has been required to equal unity. For example, in the formula of type (10) presented by Walsh (1901) the weights were defined by

$$(13) \qquad w_i = \frac{\sqrt{v_i^1 v_i^0}}{\sum\sqrt{v_j^1 v_j^0}} = \frac{\sqrt{w_i^1 w_i^0}}{\sum\sqrt{w_j^1 w_j^0}} .$$

Törnqvist (1935,1974) has used as an approximation to his integral formula a log-change index of type (10) where

$$(14) \qquad w_i = \frac{1}{2}\left(\frac{v_i^1}{V^1} + \frac{v_i^0}{V^0}\right) = \frac{1}{2}(w_i^1 + w_i^0) .$$

These weights add up to unity. As Theil (4) correctly pointed out, the choice of the weights (14) already occurred in Fisher (1922), in formula 123. The corresponding index is called in our calculations the Törnqvist Index II because of the formal correspondence there is between it and the Vartia Index II.

The Törnqvist Index I is defined by the weights (30) to be presented later.

Theil's (1973) proposal for the weights is a kind of average of Walsh's and Törnqvist's weights, the weights derived by Theil being

$$(15) \qquad w_i = \frac{T(w_i^1, w_i^0)}{\sum T(w_j^1, w_j^0)} \,, \quad \text{where}$$

$$(16) \qquad T(y,x) = \sqrt[3]{xy\left(\frac{x+y}{2}\right)} \,.$$

The mean (16) defined by Theil (which was already mentioned by Törnqvist (1935)) is a weighted geometric mean of the geometric mean $\sqrt{xy}$ and the arithmetic mean $\frac{1}{2}(x+y)$, which Theil arrived at via complicated approximations.

The properties of the means involved in the weights (12), (13), (14) and (15) are considered in Appendix 3.

By forcing the sum of the weights to equal unity, the interpretations of the log-changes (10) and (11) as arithmetic means is preserved, and the indices will then be weighted geometric means.

Yet there is no self-evident grounds for requiring that the weights in a price index number formula of form (10) should add up to unity. For example, Laspeyres's price index can be written in logarithmic form as follows

$$(17)\qquad \log\left(\frac{p^1\cdot q^0}{p^0\cdot q^0}\right) = \frac{p^1\cdot q^0 - p^0\cdot q^0}{L(p^1\cdot q^0, p^0\cdot q^0)}$$

$$= \sum \frac{q_i^0(p_i^1 - p_i^0)}{L(p^1\cdot q^0, p^0\cdot q^0)}$$

$$= \sum \frac{q_i^0 L(p_i^1, p_i^0)}{L(p^1\cdot q^0, p^0\cdot q^0)} \log(p_i^1/p_i^0) \ .$$

Thus, according to Laspeyres's formula, the weights w_i are

$$(18)\qquad w_i = \frac{L(p_i^1 q_i^0, p_i^0 q_i^0)}{L(p^1\cdot q^0, p^0\cdot q^0)} = w_i^0 \, \frac{L(p_i^1/p_i^0, 1)}{L(p^1\cdot q^0/p^0\cdot q^0, 1)} ,$$

and their sum does not necessarily equal unity. As appears from Appendix 4 (by changing only the symbols), $\Sigma w_i \leq 1$. In a corresponding manner, weights whose sum equals at most unity can be derived from Paasche's and Fisher's price indices. From this property of Fisher's index we may conclude that no bias is generally introduced although the sum of weights w_i in (10) or (11) is smaller than one, cf. Y. Vartia (1976a).

6.3. The Vartia Index II

However, if it is considered desirable for one reason or another to use an index where the sum of the weights w_i equals unity, it is best to replace in Theil's weights (15) the mean $T(y,x)$ defined by Theil with the logarithmic mean $L(y,x)$ and to define

$$(19)\qquad w_i = \frac{L(w_i^1, w_i^0)}{\sum L(w_j^1, w_j^0)} \ .$$

The index thus arrived at will be called the Vartia Index II.

These are precisely the weights that Theil (1973, 1974) and Sato (1974) endeavoured to derive. What Theil wanted to find were the weights w_i for which the sum of the logarithms of the price and volume indices would be $\log(V^1/V^0)$, as required by the factor reversal test:

$$(20) \qquad \sum w_i [\log(p_i^1/p_i^0) + \log(q_i^1/q_i^0)]$$

$$= \sum w_i \log (v_i^1/v_i^0)$$

$$= \sum w_i \log \left(\frac{w_i^1 V^1}{w_i^0 V^0}\right)$$

$$= \sum w_i \log (w_i^1/w_i^0) + (\sum w_i) \log (V^1/V^0)$$

$$= \sum w_i \log (w_i^1/w_i^0) + \log (V^1/V^0), \text{ when } \sum w_i = 1 .$$

As Theil imposed on the weights the requirement that they should add up to unity, the weights w_i have to satisfy the condition

$$(21) \qquad \sum w_i \log (w_i^1/w_i^0) \equiv 0 \ \& \ \sum w_i \equiv 1 .$$

Theil (1973) demonstrates that the condition (21) is satisfied comparatively well by Törnqvist's weights (14) when the changes in the value shares w_i^t are small, even somewhat better by Walsh's weights (13), and by Theil's own weights (15) - which he constructed on the basis of this condition - still more accurately, so that the order of smallness of the discrepancy is still increased.

For Theil's weights (15)

$$(22) \qquad \sum w_i \log (w_i^1/w_i^0) = -\frac{1}{720} \sum w_i [\log(w_i^1/w_i^0)]^5 + 0_7 ,$$

where 0_7 is a term of at least the 7th order in log-changes.

We propose to show that the weights (19) of the Vartia Index II satisfy the condition (21) identically. To this end, consider an arbitrary term of the sum (21),

$$(23) \qquad w_i \log (w_i^1/w_i^0) = w_i [\frac{w_i^1 - w_i^0}{L(w_i^1, w_i^0)}] = \frac{w_i^1 - w_i^0}{\sum L(w_j^1, w_j^0)} .$$

The sum of these terms will be zero since the numerator of the sum equals zero. The complexity of the expression in the denominator is due to the requirement $\Sigma w_i = 1$.

As I see it, the weights (19) are the only "reasonable weights" with which (21) will be satisfied identically. By "reasonable weights" I mean weights that, in the sense of an average, correspond to the value shares of the commodities[1].

6.4. Properties of our new indices

At first sight the weights (19) may seem even better than the weights (11), whose sum is less than or equal to one. Whichever of the two sets of weights is used, the indices so defined will satisfy both the time reversal test and the factor reversal test accurately

1) Sato (1975, 1976) documents an independent discovery of weights (19) and proves very interesting results using the economic theory of index numbers.

Like the weights (11), the weights (19) can be used entirely symmetrically for the prices, quantities and values, as is evident from equations (20). In this respect, then, the Vartia Indices I and II do not differ from each other.

What, then, do we lose when the sum of the weights is normed so as to equal unity, in which case the log-change in the index is interpretable as a weighted arithmetic mean?

The difference lies in the fact that the Vartia Index I is consistent in aggregation, whereas the Vartia Index II is not. As I see it, this advantage possessed by the Vartia Index I is more important than the fact that the sum of the weights equals unity, although certain formally simple and beautiful properties, reducible to the possibility of interpreting the index as an average, follows from this fact. For instance, Vartia Index II satisfies the proportionality tests PT1-PT7 whereas Vartia Index I satisfies only PT1-PT2, for a discussion of this point, see Y. Vartia (1976a).

The Vartia Index II is not consistent in aggregation, as the weights

$$(24) \qquad w_i = \frac{L(w_i^1, w_i^0)}{\sum L(w_j^1, w_j^0)}$$

are based in an excessively complicated manner on the shares

$w_i^t = v_i^t/V^t$, t = 1 or 0, computed relative to the aggregate value $V^t = v_1^t+\ldots+v_n^t$ of the entire set of commodities $A = \{a_1,\ldots,a_n\}$.

The weights of the sub-index for any subset A_k of the partition $A_1,\ldots,A_K$ of A ought to be determined exclusively by means of the value shares determined in that subset. Yet the relation between the total index and the sub-indices thus obtained is not so simple as demanded by the condition formulated in chapter 3 for consistency in aggregation.

The "proof" presented by Theil (1973) for the consistency in aggregation of Theil's index can be rendered in the following simplified form in the case of three commodities.

Consider an arbitrary index based in log-chages and let the sum of its weights w_i be equal to unity.

Consider the following partition of the set A:

$A = \{a_1,a_2,a_3\} = \{a_1,a_2\} \cup \{a_3\}$. Then

$$(25) \qquad \sum w_i \log(p_i^1/p_i^0)$$
$$= w_i \log(p_1^1/p_i^0) + w_2 \log(p_2^1/p_2^0) + w_3 \log(p_3^1/p_3^0)$$
$$= (w_1 + w_2)\left[\left(\frac{w_1}{w_1+w_2}\right) \log(p_1^1/p_1^0) + \left(\frac{w_2}{w_1+w_2}\right) \log(p_2^1/p_2^0)\right]$$
$$+ w_3 \left[\log(p_3^1/p_3^0)\right] .$$

Theil calls the weights $w_1/(w_1+w_2)$ and $w_2/(w_1+w_2)$ "conditional shares", and the expressions in brackets he calls "group indices". Equation (25) shows how the total index may be decomposed into a weighted mean of the "group indices". But the "group indices" are not sub-indices computed from the data available on the subset, since, e.g., the "conditional share" $w_1/(w_1+w_2)$ cannot generally be determined from the data on the subset $\{a_1,a_2\}$ alone. For example, when weights of type (14) are used, we have

$$(26) \qquad \left(\frac{w_1}{w_1+w_2}\right) = \frac{\frac{1}{2}(w_1^1+w_1^0)}{\frac{1}{2}(w_1^1+w_1^0) + \frac{1}{2}(w_2^1+w_2^0)} \quad , \text{ where}$$

$$(27) \qquad w_i^1 = v_i^1/(v_1^1+v_2^1+v_3^1) \text{ and } w_i^0 = v_i^0/(v_1^0+v_2^0+v_3^0) \ , \quad i = 1,2.$$

The terms v_3^1 and v_3^0 occurring in the divisiors of w_i^1 and w_i^0 cannot be determined from the data on the subset $\{a_1,a_2\}$ and they do not usually cancel out from (26). Therefore the weight (26) depends on data obtainable from the subset $\{a_1,a_2\}$. This implies that a "group index" (of clothing, say) may be changed somewhat by changing data outside the subset of the commodities in question (by changing, e.g. the quantity of meat).

In consequence, the decomposition (25) does not permit conclusions concerning the consistency in aggregation of the index. That confusion on this point is of frequent occurrence[1)] is perhaps due to the fact that so far there has been no precise mathematical definition of the property in question.

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1) For instance, Christensen & Jorgenson (1970) p. 26 maintain erroneously that the Törnqvist index defined by the weights (14) is consistent in aggregation. They must have in mind Theil's 'definition' of the consistency in aggregation, see Theil (1967) p. 159. As we briefly demonstrated, in the case of this weaker consistency in aggregation no such hierarchic system of indices as our definition in chapter 3 guarantees can be constructed.

The consistency in aggregation of the Vartia Index I can be demonstrated as follows. Consider a partition $A_1, \ldots, A_K$ of A and write

$$(28) \qquad \sum_{a_i \in A} \frac{L(v_i^1, v_i^0)}{L(V^1, V^0)} \log(p_i^1/p_i^0) = \sum_{k=1}^{K} \frac{L(V_k^1, V_k^0)}{L(V^1, V^0)} \left[\sum_{a_i \in A_k} \frac{L(v_i^1, v_i^0)}{L(V_k^1, V_k^0)} \log(p_i^1/p_i^0) \right] .$$

The bracketed expression is in fact the Vartia Index I as computed exclusively from data on the subset A_k because V_k^t equals the total value of the commodities in the set A_k during the period t = 1 or 0.

Correspondingly, the sub-indices given in brackets have been combined according to the Vartia Index I and the results equal always the Vartia Index I for A. Thus Vartia Index I can be used to construct a hierarchic system of indices starting from the subsets A_k and aggregating the subindices calculated for the subsets, the result thus obtained being consistent with the index for A.

Index number formulas of a similar type that are consistent in aggregation but do not meet the factor reversal test are easy to construct. If Walsh's weights (13) were replaced with

$$(29) \qquad w_i = \frac{\sqrt{v_i^1 v_i^0}}{\sqrt{V^1 V^0}} = \frac{G(v_i^1, v_i^0)}{G(V^1, V^0)} = G(w_i^1, w_i^0) \quad ,$$

Törnqvist's weights (14) with

$$(30) \qquad w_i = \frac{\frac{1}{2}(v_i^1+v_i^0)}{\frac{1}{2}(V^1+V^0)} = \frac{M(v_i^1,v_i^0)}{M(V^1,V^0)} \quad ,$$

and Theil's weights (15) with

$$(31) \qquad w_i = \frac{T(v_i^1,v_i^0)}{T(V^1,V^0)}$$

index number formulas consistent in aggregation would be obtained. The weights (30) possess, in addition, the property $\Sigma w_i = 1$, the other weights possessing the property $\Sigma w_i \leq 1$. The index corresponding to weights (30) is called the Törnqvist Index I in our calculations. Törnqvist (1937, p. 81) evidently meant the weights (30) rather than the weights (14) by "the share of commodity a_i in the total expenditure during the period (a,b) under consideration". (We have used here our notation.)

The indices thus obtained would react, however, in an undesirable manner to extreme price changes of individual commodities, this being the case, e.g., when the ratio p_i^1/p_i^0 approaches zero or, in other words, when a_i becomes a free good. On the realistic assumption that the quantity of a_i will remain finite even in such a situation, v_i^1 will tend to zero together with the price. This means that they do not satisfy the determinateness test of chapter 3. The Walsh-type weights (13) and (29) will then tend to zero so rapidly that the effect of the price reduction on the log-change in the entire price index

$$(32) \qquad w_i \log(p_i^1/p_i^0)$$

will tend to zero. Thus, according to the Walsh index, an extreme reduction in the price of a given commodity will not decrease the general price level at all!

On the other hand, the Törnqvist-type weights (14) and (30) will approach a positive constant, and thus the effect of the price cut, (32), will tend to minus infinity. According to the Törnqvist index, the general price level will fall down to zero together with the price of a_i!

The Theil-type weights (15) and (31) will behave qualitatively as the Walsh-type weights.

An argument in support of his choice of weights was, as Theil (1973) pointed out, that there was "no problem of infinite index changes". Yet he failed to mention that these weights would completely "kill" any extreme price cuts. Whatever nice properties according to the economic theory of index numbers Törnqvist's, Walsh's and Theil's indices may have they do not stand the common sense criterion of extreme price or volume changes.

That the way in which the Vartia Indices I and II react to large price cuts is qualitatively correct can easily be demonstrated by considering the <u>contribution</u> of a change in the value of a_i to the log-change in the aggregate value:

$$(33) \qquad \frac{L(v_i^1, v_i^0)}{L(V^1, V^0)} \left[\log(p_i^1/p_i^0) + \log(q_i^1/q_i^0)\right] .$$

Let us carry out the computations for the Vartia Index I, from which the general principle is apparent. Assuming that q_i^1 will remain finite, the weight tending to zero will 'kill' the volume term of (33) and thus, in the limit, the effect of the change in price will coincide with that of the change in value which is

$$(34) \qquad \frac{L(v_i^1,v_i^0)}{L(V^1,V^0)} \log(v_i^1/v_i^0) = \frac{v_i^1 - v_i^0}{L(V^1,V^0)} \,.$$

This will approach the value

$$(35) \qquad - \frac{v_i^0}{L(V^1,V^0)} \approx - v_i^0/V^0 = - w_i^0 \quad ,$$

when p_i^1 and, simultaneously, v_i^1 tend to zero. As appears from equation (30) of chapter 5, the Divisia-Törnqvist's general integral formula behaves qualitatively in a like manner.

An extreme reduction of the price of a commodity thus results in a decrease in the logarithm of the Vartia Index I or II approximately equal to the value share w_i^0 of the commodity. These indices consequently react as they should when a commodity disappears as a free good from the market. If conversely a free good ($p_i^0 = 0$) becomes a positively priced commodity ($p_i^1 > 0$) it will contribute approximately $+w_i^1$ to logarithm of the price index. Completely dual results hold for quantities. These are the first indices explicitly of the geometric (or the logarithmic) type having these properties[1).

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1) The aggregative type of indices, e.g., Laspeyres', Paasche's and Fisher's indices, have the same property. Their implicit representation as logarithmic indices (compare (18)) reveal the logarithmic means in their weights.

It is not difficult to construct other indices based on relative changes that will react qualitatively correctly to free goods. For example, the Walsh type of weighting (29) ought to be applied not to log-changes but to relative changes determined with respect to the geometric mean. The following equation defines then a new price index formula P_0^1:

$$(36) \qquad \frac{P_0^1 - 1}{\sqrt{P_0^1}} = \sum \frac{\sqrt{v_i^1 v_i^0}}{\sqrt{V^1 V^0}} \left[\frac{p_i^1 - p_i^0}{\sqrt{p_i^1 p_i^0}} \right]$$

Correspondingly, the Törnqvist-type weighting (30) will lead to equation

$$(37) \qquad \frac{P_0^1 - 1}{\frac{1}{2}(P_0^1 + 1)} = \sum \frac{\frac{1}{2}(v_i^1 + v_i^0)}{\frac{1}{2}(V^1 + V^0)} \left[\frac{p_i^1 - p_i^0}{\frac{1}{2}(p_i^1 + p_i^0)} \right]$$

and the Theil-type weighting (12) to

$$(38) \qquad \frac{P_0^1 - 1}{T(P_0^1, 1)} = \sum \frac{T(v_i^1, v_i^0)}{T(V^1, V^0)} \left[\frac{p_i^1 - p_i^0}{T(p_i^1, p_i^0)} \right] .$$

It is easy to show that P_0^1:s defined by (36)-(38) are price indices. These three indices are, like the Vartia Index I, consistent in aggregation and will react correctly to the occurence of free goods. Moreover, they satisfy the time reversal test but do not meet the factor reversal test.

Particularly the index defined by (37) can be recommended because of its simplicity, to replace, e.g., the Törnqvist or the Theil index. We have called it the Vartia Index III in our calculations.

The index number formulas of this type correspond to equation (5), from which we started, and are of the general form

(39) $$H(P_0^1) = \sum \frac{K(v_i^1, v_i^0)}{K(V^1, V^0)} H(p_i^1/p_i^0) \text{ , where}$$

(40) $$H(y/x) = \frac{y/x - 1}{K(y/x,\ 1)} = \frac{y - x}{K(y,x)}$$

is the indicator of relative change.[1)]

The index number formulas P_0^1 defined by (39) satisfy the time reversal test if $H(\frac{y}{x})$ is symmetric, i.e., if $K(y,x)=K(x,y)$.

The index number formulas of this type are all consistent in aggregation and will react in a qualitatively correct manner to free goods. Only when the log-change, which has been found to be the best in this respect, is chosen as the indicator of relative change (40) will an index satisfying the factor reversal test result, this index being the Vartia Index I.

It should be stated, by way of conclusion, that from the factor reversal property of the Vartia Index I, i.e.,

(41) $$\log(V^1/V^0) = \log P_0^1 + \log Q_0^1 \text{ , or}$$

(42) $$\sum w_i \log(v_i^1/v_i^0) = \sum w_i \log(p_i^1/p_i^0) + \sum w_i \log(q_i^1/q_i^0) \text{ , where}$$

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1) Another family of index numbers based on relative changes may be defined by replacing (39) by

(39') $$H(P_0^1) = \sum \frac{K(w_i^1, w_i^0)}{\Sigma K(w_j^1, w_j^0)} H(p_i^1/p_i^0).$$

This gives Vartia Index II as a special case.

$$(43) \qquad w_i = \frac{L(v_i^1, v_i^0)}{L(V^1, V^0)} = w_i^0 \frac{L(v_i^1/v_i^0, 1)}{L(V^1/V^0, 1)},$$

it follows that each terms of the sums (42) can be interpreted literally as the effect on or contribution to the log-change in the corresponding index of a change in value, price or quantity. From the consistency in aggregation of the Vartia Index I it follows that, by adding up the contributions of the commodities a_i in a given subset A_k, we will always obtain the total effects of the commodity group in question upon the changes in value, price and quantity. Time periods and prices and quantities will invariably be dealt with completely symmetrically. However, Vartia Index I satisfies only rather weak forms PT1 and PT2 of our proportionality tests. Some index theorists may regard this as a serious drawback, see however Y. Vartia (1976a).
Note also that by using the chain principle we get index series which may violate even PT1 and the weak identity test.

The most serious rival of the Vartia Index I seems to be an ingenious index suggested by Stuvel (1957) based on a symmetric decomposition of the arithmetic change in value into a price and a volume factor. The index meets both the factor reversal and the time reversal test, reacts qualitatively correctly to extreme price and volume changes and is consistent in aggregation. For a definition of Stuvel's index, see Appendix 9. If we do not demand consistency in aggregation the most serious rivals seem to be Fisher's Ideal Index and Vartia Index II using the weights (24), see also Sato (1976).

The calculations made in accordance with (42) can be illustrated graphically in the form of a "balance" by measuring the time periods along the horizontal axis and the effects of the commodities (or groups of commodities) on the changes in value, price or quantity along the vertical axis. For each commodity, three curves describing these effects will be drawn. By summation the log-changes in the value, price and volume indices (41) will be obtained from these curves. The indices are best computed by applying the chain principle, so that consecutive time periods are always compared with each other. Such calculations would furnish a well-founded alternative to, say, the present national accounting procedure, in which price and volume movements are followed by calculating the values of various commodity groups at the prices of a given base year. Regarding its accuracy, this procedure based on so-called constant-price value series is, as already noted, satisfactory at best.

7. SOME EMPIRICAL INDEX NUMBER CALCULATIONS

The aim of this chapter is to illustrate, by practical computations, quantitative differences between various formulas and different methods of calculations. The empirical data consists of Finnish yearly Gross Domestic Product (GDP) figures at factor cost by industries in 1957-72 and Finnish monthly imports of fuels and lubricants figures (cif) by the SITC subgroups in January 1972 - September 1974. The GDP data provides a fairly easy case and the results of various formulas and methods do not differ much from each other. On the other hand the imports data puts the index number formulas to a difficult test, which only the best of them are able to pass.

7.1. Finnish yearly GDP by industries

The data

As a basis of our calculations we have used yearly GDP figures at factor cost for 44 industries as they are reported in Finnish National Accounting.[1)] The values v_i^t of GDP for different

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1) Central Statistical Office of Finland (1973) table 1 and table 4 for period 1964-1972 and Central Statistical Office Of Finland (1968) table 1 and table 4 for period 1957-1964.

industries a_i measure the value added produced in those industries. The volume ratios q_i^t/q_i^{1964} were calculated using the reported official indices of production, which in turn have been calculated on the basis of the GDP series at 1954 or 1964 prices. For some industries the volume ratios for the period 1964-72 were calculated using the available GDP series at the 1964 prices to get more than the 'official' three significant figures for our volume ratios, which are given in appendix 7. The price ratios p_i^t/p_i^{1964} were defined as the ratio between v_i^t/v_i^{1964} and q_i^t/q_i^{1964}. Thus our basic data consists of values v_i^t, volume ratios q_i^t/q_i^{1964} and price ratios p_i^t/p_i^{1964}, where t means any year between 1957 and 1972. All the calculations were carried out on the computer and the data was checked up to give the correct totals and, e.g., correct figures for the official Laspeyres' volume index for GDP in 1964-72.

Index calculations

The calculations have been carried out using Laspeyres', Paasche's, Fisher's and Vartia I indices. Both base indices of the form P_{1964}^t and chain indices were calculated. The results, however, are presented both as index series (where the value of the index for 1964 is set as equal to 100) and as yearly dynamic changes[1]. These are presented in tables 1 and 2. In order to get an idea of the variation of the dynamic changes in values, prices and quantities among various industries we have calculated their variances (i.e. $10^4 s_v^2$, $10^4 s_p^2$ and $10^4 s_q^2$ in the notation of appendix 6) and the covariance of dynamic changes in prices and quantities $10^4 \operatorname{cov}(\dot{p},\dot{q})$. These statistics use the weights

1) For terminology see Introduction and Appendix 5.

of Vartia Index I. The results are presented in table 3, where also the dynamic changes in the total value of GDP $100 \log (V^t/V^{1964})$, the dynamic changes in its price $100 \log P^t_{1964}$ and in its quantity $100 \log Q^t_{1964}$ as calculated by Vartia Index I, are given. These dynamic changes satisfy

(1) $$100 \log (V^t/V^{1964}) = 100 \log P^t_{1964} + 100 \log Q^t_{1964}$$

because the factor reversal test is here satisfied.

We note that, e.g., the variance of the dynamic changes in prices $10^4 s_p^2$ is literally the variance of the price changes $100 \log (p_i^t/p_i^{1964})$ of various industries a_i around their mean $100 \log P^t_{1964}$, i.e., around the change in the price level. As is proved in appendix 6, the variances and the covariance satisfy

(2) $$s_v^2 = s_p^2 + 2\mathrm{cov}(\dot{p},\dot{q}) + s_q^2 \ .$$

The same equation applies if both sides are multiplied by 10^4. Similar variances and covariances are used by Törnqvist (1937), Rajaoja (1957) and especially Theil (1965,1967,1970), see also Y.Vartia (1976b).

Table 1. Index series for the price of GDP. The value of index for 1964 is set equal to 100, change is the dynamic change from the preceding year

Index formula Base year Chain index	Laspeyres 1964 -		Laspeyres - Chain index		Paasche 1964 -		Paasche - Chain index	
Year	Index	Change	Index	Change	Index	Change	Index	Change
1957	71.178		70.945		70.728		71.076	
1958	75.981	6.530	75.550	6.289	75.259	6.209	75.726	6.337
1959	78.336	3.052	77.856	3.006	77.730	3.231	77.992	2.943
1960	80.756	3.042	80.374	3.183	80.260	3.203	80.553	3.230
1961	84.215	4.195	83.966	4.372	83.906	4.443	84.099	4.308
1962	86.825	3.052	86.673	3.173	86.668	3.238	86.786	3.146
1963	92.975	6.843	92.922	6.962	92.922	6.968	92.976	6.889
1964	100.000	7.284	100.000	7.341	100.000	7.341	100.000	7.284
1965	104.669	4.564	104.669	4.564	104.590	4.488	104.590	4.488
1966	109.565	4.571	109.568	4.574	109.404	4.499	109.480	4.569
1967	114.836	4.698	115.023	4.859	114.849	4.858	114.913	4.844
1968	125.788	9.110	126.025	9.135	125.848	9.145	125.963	9.181
1969	133.576	6.007	133.741	5.943	133.195	5.675	133.701	5.962
1970	140.025	4.715	140.129	4.665	139.054	4.304	139.998	4.603
1971	148.579	5.930	148.715	5.946	147.241	5.721	148.522	5.931
1972*	161.451	8.308	161.343	8.151	159.625	8.076	161.124	8.124
1973*								

Table 2. Index series for the price of GDP. The value of index for 1964 is set equal to 100, change is the dynamic change from the preceding year

Index formula Base year Chain index	Fisher 1964 -		Fisher - Chain index		Vartia I 1964 -		Vartia I - Chain index	
Year	Index	Change	Index	Change	Index	Change	Index	Change
1957	70.953		71.010		70.929		71.014	
1958	75.619	6.369	75.638	6.313	75.606	6.385	75.636	6.306
1959	78.032	3.141	77.924	2.978	78.012	3.133	77.923	2.979
1960	80.507	3.123	80.463	3.206	80.494	3.133	80.461	3.206
1961	84.060	4.319	84.032	4.340	84.056	4.330	84.031	4.341
1962	86.746	3.145	86.730	3.160	86.746	3.149	86.728	3.160
1963	92.949	6.906	92.949	6.925	92.948	6.906	92.948	6.926
1964	100.000	7.312	100.000	7.312	100.000	7.313	100.000	7.313
1965	104.630	4.526	104.630	4.526	104.630	4.526	104.630	4.526
1966	109.484	4.535	109.525	4.572	109.485	4.536	109.525	4.572
1967	114.842	4.778	114.969	4.851	114.844	4.778	114.969	4.852
1968	125.818	9.128	125.995	9.158	125.806	9.117	125.996	9.158
1969	133.386	5.840	133.722	5.952	133.386	5.851	133.731	5.958
1970	139.539	4.510	140.064	4.634	139.532	4.505	140.075	4.635
1971	147.909	5.826	148.633	5.939	147.849	5.801	148.643	5.937
1972*	160.535	8.191	161.234	8.137	160.360	8.123	161.245	8.138
1973*								

Table 3. Characteristic measures of GDP calculated by Vartia Index I (base index)

Year	Value mill. mk	Change from 1964, dyn			Variance of dynamic changes				$100\Sigma w_i$
		Value	Price	Volume	Value	Price	Volume	$10^4 cov(\dot{p},\dot{q})$	
1957	10 552	-69.488	-34.349	-35.139	289.3	153.6	257.1	- 60.7	99.772
1958	11 377	-61.963	-27.964	-34.000	205.8	152.9	240.3	- 90.7	99.834
1959	12 504	-52.517	-24.831	-27.686	199.2	160.1	191.8	- 76.3	99.838
1960	14 082	-40.628	-21.698	-18.929	128.3	125.1	125.4	- 61.1	99.895
1961	15 708	-29.700	-17.368	-12.332	92.6	78.0	87.6	- 36.6	99.924
1962	16 770	-23.159	-14.219	- 8.940	48.8	40.0	44.9	- 18.1	99.960
1963	18 532	-13.166	- 7.313	- 5.853	23.7	20.1	15.0	- 5.7	99.980
1964	21 140	0.000	0.000	0.000	0.0	0.0	0.0	0.0	100.000
1965	23 145	9.059	4.526	4.533	19.2	12.8	21.6	- 7.5	99.984
1966	24 746	15.749	9.062	6.687	76.3	39.3	66.4	- 14.7	99.937
1967	26 680	23.274	13.840	9.434	171.9	75.0	94.2	+ 1.4	99.857
1968	30 064	35.214	22.957	12.257	233.0	100.4	122.1	+ 5.3	99.808
1969	34 599	49.264	28.808	20.457	305.1	153.8	204.8	- 26.7	99.753
1970	38 906	60.996	33.313	27.684	405.4	206.4	330.9	- 65.9	99.676
1971	42 220	69.172	39.102	30.070	440.6	207.4	404.2	- 85.5	99.652
1972*	48 857	83.772	47.225	36.547	679.8	249.5	635.9	-102.8	99.473
1973*									
1974*									

Discussion of the results

Let us first examine the Laspeyres and Paasche base indices. The first impression is that they do not differ much from each other, the difference being never more than two index points, which is their difference in 1972: $L_{1964}^{1972} = 161.5$ and $P_{1964}^{1972} = 159.6$. Their relative difference in dyns is 100 log (161.5/159.6) = 1.18 dyn. As a rule $L_{1964}^{t} > P_{1964}^{t}$ except in 1967 and 1968 when their difference is very small. The relative differences of L_{1964}^{t} and P_{1964}^{t} should be, according to the results of Y. Vartia (1976b), approximately equal to $\mathrm{cov}(\dot{p},\dot{q})$ for year t:

$$(3) \qquad 100 \log (L_{1964}^{t}/P_{1964}^{t}) \approx -100 \,\mathrm{cov}\,(\dot{p},\dot{q})$$

That this is a very accurate approximation is evident from the following table

Table 4: The relative difference between L_{1964}^{t} and P_{1964}^{t} and its approximation (3), dyn

year t	1956	1957	1958	1959	1960	1961	1962	1963	1964
100 log(L/P)	..	0.64	0.96	0.78	0.62	0.36	0.18	0.06	0
$-100\,\mathrm{cov}(\dot{p},\dot{q})$	..	0.61	0.91	0.76	0.61	0.37	0.18	0.06	0

year t	1972	1971	1970	1969	1968	1967	1966	1965	1964
100 log(L/P)	1.14	0.90	0.70	0.28	-0.05	-0.01	0.14	0.08	0
$-100\log(\dot{p},\dot{q})$	1.03	0.86	0.66	0.27	-0.05	-0.01	0.15	0.08	0

The covariance $cov(\dot{p},\dot{q})$ turns negative in 1967 and 1968 but gives even here a very accurate approximation. The ratio $\log(L/P)/cov(\dot{p},\dot{q})$ of these very small numbers remains even for these years close to one (namely 0.87 and 0.90 for 1967 and 1968), so that (3) gives surprisingly accurate results.

Table 4 is arranged in such a way that years equally far from 1964 are below each other. Thus we can see how the relative difference between L^t_{1964} and P^t_{1964} increases slowly, as a rule, as we move farther away from the base year 1964. The year of devaluation[1)] 1967 and the next year 1968 form exceptions. The relative differences between L^t_{1964} and P^t_{1964} are usually smaller than 1 dyn ≈ 1 %. As Paasche's price index P^t_{1964} is the official implicit price index (deflator) of GDP for $t \geq 1965$, these relative differences are just the changes that would occur if we used the equally good (or bad) formula of Laspeyres L^t_{1964} instead of the 'official' choice P^t_{1964}. This would mean, e.g., that in 1972 the official price index 159.6 would have increased by 1.14 dyn to 161.5, while at the same time the quantity index of GDP would have decreased by the same amount, 1.14 dyn. (This applies in so far as we would only change the calculation procedure of aggregating the data for the 44 industries but keep the data for these industries unchanged. Should these be recalculated as well, much greater changes would be likely to emerge. Here we regard the data for the 44 industries as given and use it only as an illustration).

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1) The Finnish mark was devalued by 27 dyn in October 1967. This means that the value of foreign currencies rose by 31 % in respect to the Finnish mark or the value of the Finnish mark was lowered by 24 %. These different figures often occasion confusion which disappears when we use some symmetric indicator of the relative change, e.g., dynamic change.

By way of a summary we conclude that the yearly relative changes in the price of GDP as calculated from the 'official' price index P^t_{1964} are on average 0.1 dyn ≈ 0.1 %-units lower than the corresponding changes as calculated from L^t_{1964}. This is a rather small but a systematic effect. For $t \leq 1964$ P^t_{1964} grows faster than L^t_{1964}.

When turning to Fisher's index and Vartia Index I as calculated by the base method we find that their difference is only a fraction of the difference between L^t_{1964} and P^t_{1964}. Because by definition

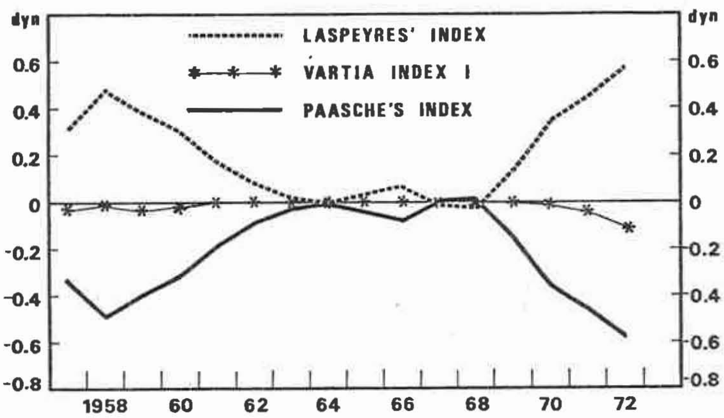
$$(4) \qquad \log F^t_{1964} = \tfrac{1}{2}(\log L^t_{1964} + \log P^t_{1964})$$

the logarithmic differences of L^t_{1964} and P^t_{1964} with respect to F^t_{1964} are equal apart from the sign:

$$(5) \qquad \log (L^t_{1964}/F^t_{1964}) = - \log (P^t_{1964}/F^t_{1964}) \approx - \tfrac{1}{2}\mathrm{cov}(\dot{p},\dot{q}) \quad .$$

These relative differences together with the relative difference between Vartia Index I and Fisher's index are shown in figure 1:

Figure 1: The relative differences of Laspeyres', Paasche's and Vartia's indices with respect to Fisher's index



We see that the Vartia Index I and Fisher's index are very accurate approximations to each other. Only in 1971 and 1972 is there a noticeable difference between them: -0.04 dyn and -0.11 dyn respectively. The differences between the yearly dynamic changes between these two indices are still smaller, as is seen from table 2. The maximum difference between these figures occurs in 1972, when it is 8.123 - 8.191 = -0.068 dyn. Usually the difference is negligible, i.e., one or two per cent of one per cent, or smaller than 0.02 dyn. On the other hand, e.g. in 1969 the yearly dynamic changes as calculated from L^t_{1964} and P^t_{1964} were 6.0 and 5.7 dyn and in 1970 4.7 and 4.3 dyn respectively. This means that in two years the price changes had been 0.7 dyn greater according to Laspeyres' index than according to the official Paasche's index. If the value of GDP had been deflated by Laspeyres' instead of Paasche's price index, the growth of 'real GDP' from 1968 to 1970 would have been 15.1 dyn instead of the official figure 15.8 dyn.

The ratios of the price indices P^k_0 and P^{k-1}_0, which are used to compare the year t_k with the year t_{k-1}, are easily derived for Laspeyres' and Paasche's indices. These are quite curious expressions showing that Laspeyres' and Paasche's price indices do not generally give reliable yearly changes in prices. For Laspeyres' index we have

$$(6) \qquad L^k_0/L^{k-1}_0 = \frac{p^k \cdot q^0}{p^{k-1} \cdot q^0} \quad .$$

Here the price comparison is based on quantities q^0 which may be badly out of date. Quantities better suitable for this comparison would have been q^{k-1} of Laspeyres' chain index.

The corresponding expression for Paasche's index reads

$$(7) \qquad P_0^k/P_0^{k-1} = \frac{p^k \cdot q^k}{p^0 \cdot q^k} \cdot \frac{p^0 \cdot q^{k-1}}{p^{k-1} \cdot q^{k-1}} \ .$$

Here the result depends not only on p^k, p^{k-1}, q^k and q^{k-1} but also on p^0.

If the prices for the years t_k and t_{k-1} remained the same, (7) would not generally be unity but would depend, e.g., on p^0. This shows that the expression (7) cannot be regarded as a fair price index from t_{k-1} to t_k as is noted, e.g., by v. Hofsten (1952) p. 37.

Like Lowe's index treated in chapter 3 , (6) and (7) interpreted as formulas of the type P_{k-1}^k do not satisfy the commodity reversal or the unit of measurement test in their strict form because of their dependance on q^0 and p^0. They are not d.c. index number formulas.

Thus we should not actually say that the prices of GDP rose by 4.3 dyn from 1969 to 1970 as measured by Paasche's price index. In fact we have not compared 1969 and 1970 directly but via 1964, and literally we should say that in 1970 the price level as compared to 1964 (i.e. 139.1) is 4.3 dyn higher than the price level in 1969 as compared to 1964 (i.e. 133.2).

Note that it is by no means necessary for a precision formula to lie between Laspeyres' and Paasche's 'limits' if these happen to be near each other. We have discussed this problem in chapter 2 . In 1967 and 1968 Laspeyres' and Paasche's price indices are almost equal. Vartia Index I lies, however, between them, being thus a better approximation to Fisher's index than neither of these two in our GDP data.

We turn next to the chain indices. As will be seen from tables 1 and 2, the chain indices calculated from the yearly changes P_{t-1}^{t} using Laspeyres', Paasche's, Fisher's and Vartia's formulas conform better with each other than do the base indices. Fisher's index and Vartia Index I differ at most in the second decimal place, their greatest relative difference being less than 0.01 dyn ≈ 0.01 %. Laspeyres' and Paasche's chain indices differ about 10 times as much, or in the first decimal place, their relative difference being always smaller than 0.2 dyn ≈ 0.2 %. The relative difference between the base indices of Laspeyres and Paasche were almost 10 times as large.

Thus, when the chain principle was used instead of the more common base principle, the different formulas gave a more uniform picture of the price development of GDP. This is because the relative importance of different industries does not have time to change substantially when consecutive years are compared with each other.

Frisch (1936) notes that Laspeyres' chain index usually has a tendency to drift upwards with respect to the base index as Paasche's index drifts downwards. In our calculations no such tendencies are manifested. While Laspeyres' chain index gives a relatively good approximation to Laspeyres' base index, Paasche's chain index seems to drift upwards rather than downwards.

Although the chain indices conform well with each other, they do not conform equally well with the base indices. These two strategies of index calculation give slightly but systematically different results. This is seen clearly from the following table 5, where we have compared the chain and base indices as calculated by the Vartia Index I.

Table 5: Relative differences between base and chain indices, $100\epsilon(t)$, and differences between yearly dynamic changes of chain and base indices, $100[\epsilon(t)-\epsilon(t-1)]$, as calculated by Vartia Index I, dyn

year t	1956	1957	1958	1959	1960	1961	1962	1963	1964
$100\epsilon(t)$	..	0.120	0.040	-0.106	-0.041	-0.030	-0.021	0.000	0.000
$100[\epsilon(t)-\epsilon(t-1)]$	..	..	-0.079	-0.154	0.073	0.011	0.011	0.020	0.000

year t	1972	1971	1970	1969	1968	1967	1966	1965	1964
$100\epsilon(t)$	0.550	0.536	0.388	0.258	0.151	0.109	0.037	0.000	0.000
$100[\epsilon(t)-\epsilon(t)]$	0.015	0.136	0.130	0.107	0.041	0.074	0.036	0.000	0.000

Here $\epsilon(t)$ is the logarithmic difference between the chain index $\bar{P}^t_{1964}$ and the base index P^t_{1964},

$$(8) \qquad \epsilon(t) = \log(\bar{P}^t_{1964}/P^t_{1964}) \quad ,$$

and $\epsilon(t)-\epsilon(t-1)$ is the logarithmic difference between the yearly changes in the chain and base indices:

$$(9) \qquad \epsilon(t)-\epsilon(t-1) = \log(\bar{P}^t_{1964}/P^t_{1964})-\log(\bar{P}^{t-1}_{1964}/P^{t-1}_{1964})$$

$$= \log(\bar{P}^t_{1964}/\bar{P}^{t-1}_{1964})-\log(P^t_{1964}/P^{t-1}_{1964}) \quad .$$

We note that the chain index $\bar{P}^t_{1964}$ drifts upwards as compared with the base index P^t_{1964}. Their relative difference is positive or zero, $\bar{P}^t_{1964} \geq P^t_{1964}$, except in 1959-1962. In 1971-72 the chain index is more than 0.5 dyn higher than the base index.
This means that the average change in prices from 1964 would be slightly higher if it were calculated via the yearly changes

P^t_{t-1} instead of using the direct comparison P^t_{1964}. All the chain indices and especially the figures calculated by Fisher's and Vartia I formulas approximate the Divisia-Törnqvist's integral formula (26) in chapter 5, where τ equals 1 year. This means that if we had continuous records on values, priced and quantities of the GDP for the 44 industries and if yearly moving averages of these were calculated before using Divisia-Törnqvist's formula, then this index series would conform very closely with, e.g., Vartia Index I in the calendar years 1957-72. Of course, GDP is not recorded 'continuously' (as is, e.g., the consumption of electric power) but only quaterly in Finland. We could use the quarterly series of GDP, calculate their one-year moving averages and get an even more accurate approximation to the integral formula. Because changes in one-year moving averages are probably very slow and smooth, these calculations would give practically the same results as our chain index calculations based on figures from consecutive calendar years.

If no moving averages of the quaterly figures were calculated but the chain indices were computed by comparing consecutive quarters to each other, quite different results would have been obtained. These chain indices would approximate the hypothetical results given by the Divisia-Törnqvist's integral formula with τ equal to one quarter of a year. These indices compare prices of different quarters to each other. By calculating yearly (moving) averages of these quaterly index series comparisons of a kind of yearly prices could be made. Because the calculation of moving averages and that of chain indices do not usually commute as mathematical operations, the order in which they are applied does matter.

Finally we comment on the calculations presented in table 3. Here we have given dynamic changes of, e.g., the price index $100 \log P^t_{1964}$ instead of the ordinary index series $100\, P^t_{1964}$ or the percentual changes $100(P^t_{1964}-1)$. Here the multiplicative identity between the value ratio and the price and volume indices, $V^t/V^{1964} = P^t_{1964}\, Q^t_{1964}$, is transformed into an additive identity between their dynamic changes.

For instance, in making the comparison 1964→1971 we find that the dynamic change in the value of GDP, 69.172, which approximately corresponds to an increase of 100 %, consists of a rise of 39.102 dyn in the price level and an average increase of 30.070 dyn in the quantities. These figures are mean values of a kind of the dynamic changes in values, prices and quantities in the various 44 industries. These individual dynamic changes are not even approximately equal. For instance, the variance of the dynamic value changes, $100 \log (v_i^t/v_i^{1964})$, was 440.6 and thus the standard deviation of these dynamic value changes around their mean (69.172 dyn) was 21.0 dyn.

Likewise, the variance of the dynamic price changes $10^4 s_p^2$ equals 207.4 square dyn or $100 s_p = 14.4$ dyn. This is considerably less than the standard deviation of the quantity changes $100 s_q = 20.1$ dyn corresponding to the the variance of the quantity changes $10^4 s_q^2 =$ 404.2 square dyn. The covariance term $10^4 \text{cov}(\dot{p},\dot{q}) = -85.5$ square dyn shows that positive (negative) price deviations $\log(p_i^{1971}/p_i^{1964}) - \log P^{1971}_{1964}$ are usually connected with negative (positive) quantity deviations $\log(q_i^{1971}/q_i^{1964}) - \log Q^{1971}_{1964}$. These covariance terms $10^4 \text{cov}(\dot{p},\dot{q})$ are intimately associated with Laspeyres' and Paasche's indices.

The variances of dynamic changes allow us to compare quantitatively the differences in the price and quantity structures between various periods, see Theil (1967) p. 155. In 1957, which is like 1971 at seven years' distance from 1964, the value of GDP was 69.488 dyn smaller than (or about half of) the value of GDP in 1964. Thus the relative changes 1957→1964 and 1964→1971 were approximately equal. Neither did the dynamic changes in prices or quantities for those periods differ substantially. On the other hand the variance of the value changes, 289.3 square dyn, in 1957 is clearly smaller than the variance of the value changes, 440.6, in 1971. The difference is explained by the more uniform growth of production in the period 1957-64, shown by the variance of the quantity changes, 257.1 dyn. This is considerably smaller than the corresponding variance of the quantity changes in the period 1964-71, which is 404.2 square dyn.

By analysing symmetrically the years preceding and following 1964 we find that the price variances increase in both cases almost in the same way, see table 6.

Table 6. Variances of dynamic changes from 1964 as calculated using the weights of Vartia Index I, square dyn

year	1957	1958	1959	1960	1961	1962	1963	1964
$10^4 s_v^2$	289.3	205.8	199.2	128.3	92.6	48.8	23.7	0.0
$10^4 s_p^2$	153.6	152.9	160.1	125.1	78.0	40.0	20.1	0.0
$10^4 s_q^2$	257.1	240.3	191.8	125.4	87.6	44.9	15.0	0.0
$10^4 \mathrm{cov}(\dot{p},\dot{q})$	-60.7	-90.7	-76.3	-61.1	-36.6	-18.1	-5.7	0.0

year	1971	1970	1969	1968	1967	1966	1965	1964
$10^4 s_v^2$	440.6	405.4	305.1	233.0	171.9	76.3	19.2	0.0
$10^4 s_p^2$	207.4	206.4	153.8	100.4	75.0	39.3	12.8	0.0
$10^4 s_q^2$	404.2	330.9	204.8	122.1	94.2	66.4	21.6	0.0
$10^4 \mathrm{cov}(\dot{p},\dot{q})$	-85.5	-65.9	-26.7	+5.3	+1.4	-14.7	-7.5	0.0

We have already noted the abnormal behaviour of the covariance term in 1967-68, which may reflect the 1967 devaluation. In these years the price changes exceeding the average price changes from 1964 did not seem to have their usual reducing effect on the growth rates of production of the industries. Therefore, the variances of the value changes were considerably higher in 1967 and 1968 than in 1961 and 1960, when the covariance terms were negative, see equation (2). But in 1970 and 1971 the interdependence of price and quantity changes, as measured by their covariance, had attained its 'normal magnitude'. On the other hand in 1970 the variance of the quantity changes becomes higher than in 1958. It seems that at the beginning of the 1970s exceptional deviations had arisen in the growth rates of output, amounting to a rapid structural change of production.

Our analysis has thus far been based on the descriptive approach. An attempt could be made, however, to interpret the results yielded by the various formulas and different methods of calculation in terms of the economic theory of index numbers, as presented in chapter 2. Yet we will make no such attempt here, since it would presuppose introducing the assumption that our data would have been generated in accordance with the classic theory of time invariant demand or/and production. There are, however, problems originating from the value added character of our GDP data that make us put these interesting interpretations aside in this connection.

7.2. Finnish monthly imports of fuels and lubricants

The data

Our data is obtained from the monthly bulletins of the Board of the Customs (1972, 1973, 1974), table 2, from which we have taken the cumulated monthly sums of values and quantities of 8 SITC groups or subgroups. The monthly figures calculated from these cumulative sums and the partition used are presented in appendix 8. Quality changes are likely to have some effect on our results but these problems will be completely ignored here. We take here - as in our GDP example - the data as given and use it as an illustration only. We want to illustrate the behaviour of various formulas when the data is given, without discussing the usually relevant problems of quality changes and lack of data. Monthly series showing exceptionally great and heterogeneous price changes were selected in order to put the various index formulas to a difficult test.

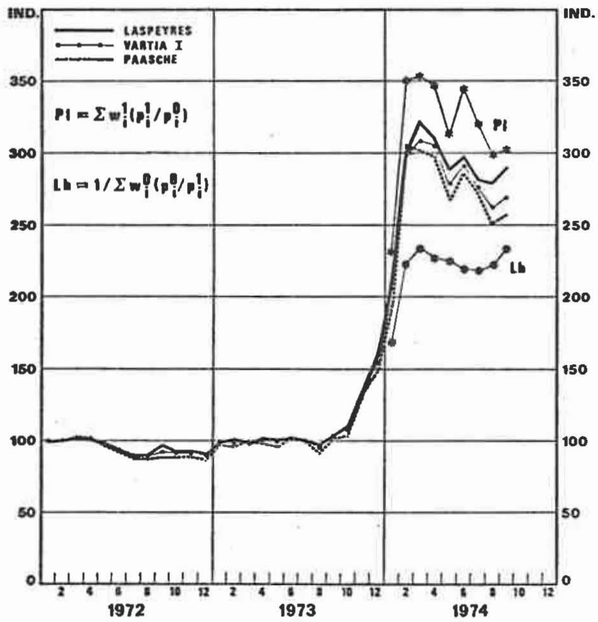
Index calculations

Only base indices with January 1972 as the base period were calculated. The formulas of Laspeyres, Paasche, Fisher, Stuvel, Vartia I-III, Törnqvist I-II and Theil were employed and the results are given in table 7. The formulas are presented in appendix 9. The calculations were carried out using a computer, where the data was stored in a memory, and this made calculation errors improbable. (Printing errors, etc. are not excluded.) To facilitate the comparison of the formulas their relative differences with respect to Fishers formula were calculated, as shown in table 8.

Discussion of the results

As will be seen from table 7 Laspeyres' index is usually greater than Paasche's index, the other indices lying between these two. In 1974/9 Laspeyres' index was 290.3 and Paasche's index was 258.0, or 11.8 dyn smaller. Differences of such a magnitude are not without relevance if, e.g., some contracts have an index clause. The other indices ranged from 267.8 to 280.5. The overall development of the import prices of fuels and lubricants is shown[1] in figure 3: the prices rose threefold in four months early in 1974.

Figure 3: Price index series for imports of fuels and lubricants. The value of the indices in January 1972 is set equal to 100



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1) Figure 3 also shows the development of Palgrave's and Harmonic Laspeyres' indices in 1974. These have here considerable biases respectively up and down, see Y. Vartia (1976b). These two index number formulas are included only in this figure to remind that the choice of an index formula may be of considerable significance.

Table 7. Index series for import price of fuels and lubricants, base indices, base period January 1972

Index formula / period		Laspeyres	Paasche	Fisher	Stuvel	Vartia I	Vartia II	Vartia III	Törnqvist I	Törnqvist II	Theil
1972	1	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
	2	100.849	100.342	100.595	100.656	100.685	100.714	100.541	100.531	100.596	100.721
	3	102.110	102.962	102.535	102.406	102.180	102.450	101.135	101.163	101.842	102.474
	4	101.363	103.020	102.188	102.062	102.062	102.162	101.789	101.751	101.993	102.163
	5	98.853	96.897	97.870	97.893	97.943	97.934	97.956	97.901	97.910	97.934
	6	94.291	92.900	93.593	93.501	93.773	93.642	93.591	93.504	93.448	93.646
	7	90.525	88.871	89.694	89.476	90.003	89.894	89.900	89.800	89.749	89.896
	8	90.374	87.891	89.124	88.952	89.517	89.381	89.340	89.249	89.253	89.379
	9	97.582	89.349	93.375	92.986	92.923	92.874	92.738	92.788	92.833	92.876
	10	92.755	88.916	90.815	90.441	91.363	91.247	91.030	90.952	91.041	91.254
	11	92.967	89.579	91.257	90.963	91.582	91.460	91.275	91.214	91.254	91.463
	12	91.452	87.378	89.392	89.166	90.047	89.877	89.779	89.680	89.679	89.844
1973	1	99.951	98.852	99.400	99.346	99.570	99.541	99.619	99.560	99.484	99.542
	2	100.196	96.505	98.333	98.736	98.743	98.683	98.687	98.642	98.509	98.691
	3	97.505	99.916	98.703	98.529	99.256	99.514	96.644	95.917	96.803	99.754
	4	102.077	98.794	100.422	100.519	100.590	100.595	100.703	100.630	100.642	100.595
	5	101.232	96.847	99.015	99.200	99.235	99.207	99.358	99.285	99.221	99.206
	6	102.035	103.508	102.769	102.705	102.744	102.860	102.423	102.365	102.499	102.870
	7	101.274	100.135	100.703	100.644	100.490	100.470	100.775	100.731	100.645	100.452
	8	98.268	92.257	95.379	94.856	95.743	95.740	95.627	95.516	95.682	95.730
	9	104.235	102.524	103.376	103.304	103.883	103.940	103.971	103.893	103.726	103.940
	10	111.590	104.535	108.005	107.496	108.292	108.526	108.247	108.208	108.429	108.529
	11	137.132	130.526	133.788	133.383	134.664	135.521	135.198	135.830	134.410	135.596
	12	160.838	149.723	155.181	156.045	154.919	156.346	155.419	156.874	155.474	156.378
1974	1	212.329	197.678	204.873	205.921	206.434	207.729	207.113	211.640	204.955	207.880
	2	300.019	306.612	303.298	301.780	299.785	304.548	305.776	317.692	298.907	304.868
	3	323.259	302.377	312.644	318.813	309.252	314.273	306.498	320.004	309.050	314.462
	4	310.941	298.068	304.436	307.658	305.359	306.043	302.597	315.116	301.064	306.204
	5	289.320	269.490	279.229	284.245	278.385	281.301	274.796	284.840	277.343	281.407
	6	298.707	287.793	293.199	295.638	292.136	294.360	293.084	307.161	290.523	294.460
	7	282.285	272.330	277.263	279.141	276.110	279.871	277.668	288.731	275.333	279.999
	8	279.552	252.393	265.625	269.453	262.330	269.675	263.726	273.527	265.970	269.869
	9	290.265	258.046	273.682	280.493	269.668	275.790	267.820	277.819	273.168	275.824
	10										
	11										
	12										

The relative differences of various indices with respect to Fisher's index are presented in table 8. The table allows comparison between all the indices included. The relative differences are about ten times as large as in our GDP example, where the corresponding differences did not exceed 0.6 dyn.

Table 8. Relative differences of index number formulas with respect to Fisher's formula: 100 log(I/F), dyn

Index formula / period		Laspeyres	Stuvel	Vartia I	Vartia II	Vartia III	Törnqvist I	Törnqvist II
1972	1	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	2	0.25	0.06	0.09	0.12	-0.05	-0.06	0.00
	3	-0.42	-0.13	-0.35	-0.08	-1.37	-1.35	-0.68
	4	-0.81	-0.12	-0.12	-0.03	-0.39	-0.43	-0.19
	5	1.00	0.02	0.07	0.07	0.09	0.03	0.04
	6	0.74	-0.10	0.19	0.05	-0.00	-0.10	-0.16
	7	0.92	-0.24	0.34	0.22	0.22	0.12	0.06
	8	1.39	-0.19	0.44	0.29	0.24	0.14	0.14
	9	4.41	-0.42	-0.49	-0.54	-0.68	-0.63	-0.58
	10	2.11	-0.41	0.60	0.47	0.24	0.15	0.25
	11	1.86	-0.32	0.36	0.22	0.02	-0.05	-0.00
	12	2.28	-0.25	0.73	0.54	0.43	0.32	0.32
1973	1	0.55	-0.05	0.17	0.14	0.22	0.16	0.08
	2	1.88	0.41	0.42	0.36	0.36	0.31	0.18
	3	-1.22	-0.18	0.56	0.82	-2.11	-2.86	-1.94
	4	1.63	0.10	0.17	0.17	0.28	0.22	0.23
	5	2.21	0.19	0.22	0.19	0.35	0.27	0.21
	6	-0.72	-0.06	-0.02	0.10	-0.34	-0.39	-0.26
	7	0.57	-0.06	-0.21	-0.23	0.07	0.03	-0.06
	8	2.98	-0.55	0.38	0.38	0.12	0.14	0.32
	9	0.83	0.07	0.49	0.55	0.58	0.50	0.34
	10	3.27	-0.47	0.27	0.48	0.22	0.18	0.39
	11	2.47	-0.30	0.65	1.29	1.05	1.51	0.46
	12	3.58	0.56	-0.17	0.75	0.15	1.09	0.19
1974	1	3.57	0.51	0.76	1.38	1.09	3.25	0.04
	2	-1.09	-0.50	-1.17	0.41	0.81	4.64	-1.46
	3	3.34	1.95	-1.09	0.52	-1.99	2.33	-1.16
	4	2.11	1.05	0.30	0.53	-0.61	3.45	-1.11
	5	3.55	1.78	-0.30	0.74	-1.60	1.99	-0.68
	6	1.86	0.83	-0.36	0.40	-0.04	4.65	-0.92
	7	1.80	0.68	-0.42	0.94	0.15	4.05	-0.70
	8	5.11	1.43	-1.25	1.51	-0.72	2.93	0.13
	9	5.88	2.46	-1.48	0.77	-2.17	1.50	-0.19
	10							
	11							
	12							

For instance, Laspeyres' index for 1972/9, 1974/8 and 1974/9 was more than 4 dyn higher than Fisher's index. Only for five months, namely 1972/3, 1972/4, 1973/3, 1973/6 and 1974/2, was it below Fisher's index. Paasche's index is not presented in table 8, because we have simply log(P/F) = - log(L/F). Figure 4 shows that Vartia Indices I and II are very accurate approximations to each other up to 1973/10 when prices started to rise fast. Thereafter Vartia II > Vartia I. Both of them approximate Fisher's index accurately all the time, being as a rule closer to it than Laspeyres' and Paasche's indices, which usually deviate about three times as much from Fisher's index. In our material Vartia II seems to have a slight tendency toward exceeding Fisher's index.

Figure 4. Relative differences of index number formulas with respect to Fisher's formula: 100 log(I/F)

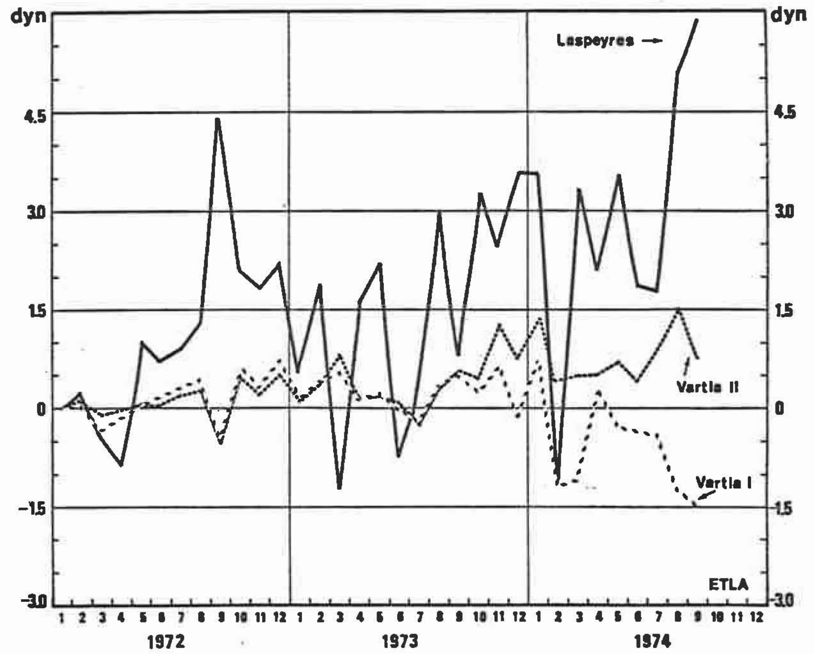
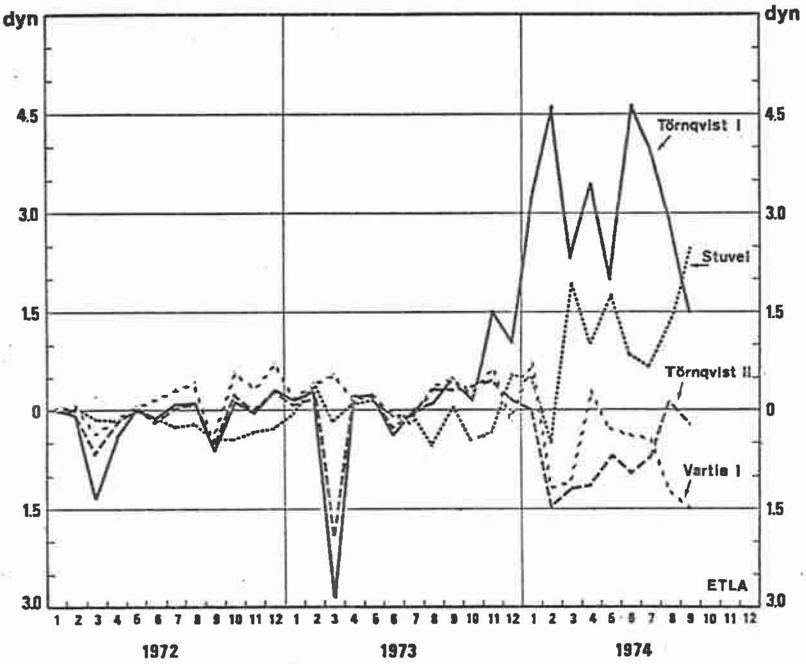


Figure 5. Relative differences of index number formulas with respect to Fisher's formula: 100 log(I/F)



From figure 5 we see that up to 1973/10 Törnqvist Indices I and II are very near each other. That this should be so is clear from their definitions given in appendix 9: they are both means of the 'logarithmic Laspeyres index' log l and the 'logarithmic Paasche index' log p discussed in Y. Vartia (1976b). The logarithm of the Törnqvist Index I is their weighted average

(10) $$\Theta \log p + (1 - \Theta)\log l = \Theta \Sigma w_i^1 \log(p_i^1/p_i^0) + (1 - \Theta)\Sigma w_i^0 \log(p_i^1/p_i^0) \, ,$$

where $\Theta = V^1/(V^1 + V^0)$, while

the logarithm of the Törnqvist Index II is their 'unweighted' average

(11) $$\log t = \tfrac{1}{2}(\log p + \log l) \, .$$

These are, of course, almost equal if $\log p \approx \log 1$ or $\Theta \approx 1/2$. When prices started to rise rapidly, Θ increased and Törnqvist Index I began to exceed Törnqvist Index II, since here we have $\log p > \log 1$. We are not going to give a thorough analysis of the situation in the spirit of Y. Vartia (1976b) but will only mention that the situation is somewhere between Fisher's five-tined fork and our new five-tined fork, where $\log Pl > \log p \approx \log l > \log F \approx \log t > \log P \approx \log l > \log Lh$. The analysis would be based on the quantities $\mathrm{cov}(\dot{p},\dot{q})$ and $\mathrm{cov}(\dot{p},\dot{v})$. Usually the Vartia Indices I and II and the Törnqvist Indices I and II approximate each other very accurately. Their relative differences are often smaller compared with each other than with Fisher's index. They begin, however, to deviate from each other after 1973/10. In 1973/3 the Törnqvist Indices I and II give clearly too small values compared, e.g., with Fisher's index. We shall discuss this phenomenon presently in detail.

Stuvel's index gives a very good approximation to Fisher's index up to the beginning of 1974. Its course is not, however, the same in detail as that of the former indices, which are closely related to each other. In 1974 its deviations from Fisher's formula begin to resemble those of Laspeyres' formula, being about half of them in magnitude. As may be proved, Stuvel's formula approaches Laspeyres' formula if all the price ratios p_i^1/p_i^0 tend to infinity while the volume ratios remain approximately constant, as happened in our imports data.

In the same situation Törnqvist Index I given in (10) approaches logp as does Vartia Index I. Törnqvist Index I shows this tendency clearly but Vartia Index I seems to move in the opposite direction in 1974. This shows that the asymptotic behaviour of Vartia Index I is not revealed if prices have 'only' grown threefold. This is a consequence of the utmost slowness with which the logarithmic mean L(x,y) approaches zero as x or y approaches zero, see appendix 3. The qualitatively correct behaviour with respect to extreme price changes of the Vartia Indices I and II result from the same circumstance.

From table 8 the Vartia Index III is seen to be almost equal to the Törnqvist Index I up to 1973/10. This is what we expected from its definition, see equation (37) in chapter 6. After that point this index follows a peculiar course of its own.

From table 7 we see that Theil's formula (15) in chapter 6 so closely approximates the Vartia Index II that we have excluded it from table 8. The difference between the Theil and Vartia II indices is usually met in the second decimal place. Even in the difficult month 1973/3 Theil's Index is only about 0.2 dyn higher than the Vartia Index II.

In this period especially Törnqvist's indices and Vartia Index III deviate considerably from the indices yielded by other precision formulas. It is evident that these three indices are here in error. Examination of our data reveals the reason for their behaviour. We have collected the relevant data for 1972/1 or t_0 and 1973/3 or t_1 in the following table.

Table 9. Imports of fuels and lubricants in 1972/1 and 1973/3 by SITC subgroups

SITC no.	1972/1 q^0	v^0 mill.mk	w^0 %	1973/3 q^1	v^1 mill.mk	w^1 %	p^1/p^0 %
321.4	92.6	8.1	6.0	181.2	9.6	9.8	60
321.8	85.6	18.4	13.7	11.1	0.9	0.9	36
331	781.4	69.0	51.3	556.8	54.2	55.3	110
332.1	0.6	0.1	0.1	0.3	0.1	0.1	151
332.2	7.5	1.0	0.7	2.4	0.4	0.4	123
332.3	168.4	27.0	20.0	111.9	20.1	20.5	108
332.4	69.3	5.6	4.2	113.7	9.4	9.6	102
332.5	12.5	5.4	4.0	5.5	3.3	3.4	139
Total	.	134.6	100.0	.	97.9	100.0	.

As is evident from table 9 the problem is caused by SITC subgroup 321.8 (coke and semicoke), for which the value of imports decreased from 18.4 mill.mk to 0.9 mill.mk when its unit value (price) fell to 36 % of the previous unit value. Its value share fell from 13.7 % to 0.9 %, so that it practically disappeared from the import market when both its quantity and price fell sharply.

Since both of Törnqvist's formulas react quantitatively wrongly to extreme price cuts[1)] it is nothing of a surprise that they give too low values for 1973/3. The Vartia Indices I and II and, e.g., Fisher's and Stuvel's indices react qualitatively correctly to extreme price cuts, and therefore they do not show any strange behaviour in this period. The Vartia Index III follows, to our surprise, closely the Törnqvist Index I although it reacts qualitatively in a right way to extreme

1) Divisia-Törnqvist index in general integral form as given in chapter 5 behaves qualitatively correctly here although these two of its discrete approximations do not.

price cuts; at least it doesn't become zero together with any of the price ratios as, e.g., the Törnqvist Indices do. The contribution of a price cut to the Vartia Index III is, however, twice as large as would be appropriate and it apparently begins to be felt too early, i.e., even for quite moderate price cuts.

Although the Theil Index seems to behave correctly in 1973/3 it is unsensitive to extreme price cuts. Something of this kind of behaviour can be seen from the figures in table 8: Theil's index exceeds the Vartia Index II by 0.2 dyn just in 1973/3.

We conclude that only the formulas of Fisher, Vartia I and II, Theil and Stuvel give results which cannot be maintained to be in error. All these formulas, except the formula of Theil, are 'ideal index numers' satisfying, e.g., the factor reversal test. There are situations, as we have noted, where Theil's formula gives unacceptable results, but our data did not reveal such. Here Theil's formula is a very accurate approximation to the Vartia Index II.

8. SUMMARY

We started from mathematically elementary but empirically important problems connected with various indicators of relative change. Our considerations led us to recommend the log-change $H_4(y/x) = \log(y/x) = \ln(y/x)$ as the most suitable indicator of the relative change from x to y and suggest such new terminology as we have used throughout the text.

Our approach to the index number problem may be characterized as statistical or descriptive one, as opposed to the economic theory of index numbers. These various approaches were discussed in chapter 2.

Chapter 3 provided an axiomatically oriented treatment of the descriptive theory of price and quantity index formulas in the spirit of Fisher (1922). Various desiderata (or, in Fisher's terminology, 'tests') were presented and discussed. These desiderata included 'consistency in aggregation', which was formulated in exact terms. This concept has previously been only vaguely defined, which state of affairs has apparently caused unnecessary confusion.

Different strategies of index series construction, as opposed to the choice of the index formula, were distinguished and discussed in chapter 4. The chain principle leads naturally to the continuous approach of Divisia and Törnqvist, which was critically reviewed in chapter 5. Problems inherent in the definition of continuous values, quantities and prices in time were particularly discussed. It was shown that the customary definitions of Divisia-Törnqvist's indices ignore the effect of the smoothing parameter τ (or some other similar parameter) on the definition of the indices.

In chapter 6 we derived (among other formulas) two new 'ideal log-change index number formulas'. The second of them, the Vartia Index II, amounts to the solution of a problem presented in Theil (1973). Its discovery, early in 1974, was stimulated by Theil (1973), who together with Sato (1974) derived good approximative solutions to this problem, which Theil (1974) was already inclined to regard as unsolvable. Sato (1975, 1976) has afterwards independently discovered the same (and only) solution to the problem. Our first index, the Vartia Index I, already discovered in 1973, is similar to our second index. Unlike the Vartia Index II, it possesses the interesting and often desirable property of consistency in aggregation but its behaviour under proportional changes in prices or quantities is more complex than the behaviour of the Vartia Index II, see also Y. Vartia (1976a). Both of our new indices react qualitatively correctly to extreme price and quantity changes (i.e., satisfy the determinateness test), satisfy the time and factor reversal tests and give excellent approximations as chain indices to Divisia-Törnqvist's integral formula. Their approximation properties are only briefly discussed in this monograph, but they may be derived using methods of Y. Vartia (1976b). Diewert (1975) presents some results of the approximation properties of Vartia Index I, and here some future research seems promising.

Leo Törnqvist has proposed in a discussion with the author an interesting definition of a one-parameter family of index number formulas which yields our two new indices as special cases. Törnqvist's definition is analogous to the one given by van Yzeren (1958) for Stuvel's index and provides another interesting object of further study. The construction and properties of the Vartia Indices I and II may offer useful new insight into, e.g., the demand and production theories[1] and the aggregation of economic relations[2], which were deliberately ignored in this monograph.

The numerical results obtained by using different formulas and base or chain principles in constructing index series were illustrated by two empirical examples in chapter 7. The relative differences between our new index formulas and various other formulas were analysed and briefly explained. Even in the exceptionally difficult material of monthly imports of fuels and lubricants during the oil crisis, when prices rose threefold, the Vartia Indices I and II and, e.g., Fisher's ideal index remained inside a band having a height of 3 dyn ≈ 3 %. Laspeyres' and Paasche's indices deviated from each other about four times as much. Our two new index formulas were thus found to behave in accordance with the theoretical findings of previous chapters.

Recent years have witnessed, it seems, a revival of interest in the index number problems and especially in the chain index methods[3]. It may be that, e.g., the volume series of national

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1) See e.g. Theil (1965, 1967, 1970) and Barten (1964).

2) See e.g. Gorman (1959), Pollak (1972), Blackorby, Nissen, Primont and Russel (1974), Morishima and others (1973), Muellbauer (1975).

3) See, e.g. Christensen and Jorgenson (1970), Samuelson and Swamy (1974) and Diewert (1976b) and the literature referred to by them.

accounts based on Laspeyres' quantity index formula, which have now served us for a few decades, will gradually be replaced with more accurate chain index calculations.

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Appendix 1. The identity (21) of Chapter 1.

(1) $$H_3(\tfrac{z}{x}) = H_3(\tfrac{y}{x}) + H_3(\tfrac{z}{y}) + \epsilon, \text{ where}$$

(2) $$\epsilon = -\tfrac{1}{4}H_3(\tfrac{y}{x})H_3(\tfrac{z}{y})H_3(\tfrac{z}{x}).$$

Let us start from equation (16)

(3) $$H_3(\tfrac{z}{x}) = \frac{y-x}{\frac{1}{2}(x+z)} + \frac{z-y}{\frac{1}{2}(x+z)}$$

$$= H_3(\tfrac{y}{x})+H_3(\tfrac{z}{y}) + \left[\frac{y-x}{\frac{1}{2}(x+z)} - \frac{y-x}{\frac{1}{2}(x+y)}\right] +$$

$$+ \left[\frac{z-y}{\frac{1}{2}(x+z)} - \frac{z-y}{\frac{1}{2}(y+z)}\right] .$$

(4) $$\epsilon = (y-x) \left[\frac{(x+y)-(x+z)}{\frac{1}{2}(x+z)(x+y)}\right] + (z-y) \left[\frac{(y+z)-(x+z)}{\frac{1}{2}(x+z)(y+z)}\right]$$

$$= (y-x)(y-z) \left[\frac{1}{\frac{1}{2}(x+z)(x+y)}\right] +$$

$$(z-y)(y-x) \left[\frac{1}{\frac{1}{2}(x+z)(y+z)}\right]$$

$$= (y-x)(z-y)\left[\frac{1}{\frac{1}{2}(x+z)(y+z)} - \frac{1}{\frac{1}{2}(x+z)(x+y)}\right]$$

$$= (y-x)(z-y)\left[\frac{x-z}{\frac{1}{2}(x+z)(y+z)(x+y)}\right]$$

$$= -\frac{y-x}{(x+y)} \cdot \frac{z-y}{(y+z)}\,\frac{z-x}{\frac{1}{2}(x+z)}$$

$$= -\tfrac{1}{4}H_3(\tfrac{y}{x})H_3(\tfrac{z}{y})H_3(\tfrac{z}{x}) \ .$$

By virtue of symmetry, ϵ must vanish when $H_3(\frac{z}{x}) = 0$, irrespective of $H_3(\frac{y}{x})$ and $H_3(\frac{z}{y})$. Likewise, ϵ must vanish when one or the other of these two is zero. In consequence, ϵ is a multiplicative expression in the relative changes concerned; neither the powers of these nor the possible other terms are yet determined thereby.

Appendix 1. The identity (24) of Chapter 1.

(1) $H_6(\frac{z}{x}) = H_6(\frac{y}{x}) + H_6(\frac{z}{y}) + \epsilon$, where

(2) $$\epsilon = \tfrac{1}{2}H_6(\tfrac{y}{x})H_6(\tfrac{z}{y})H_6(\tfrac{z}{x})\left[\frac{1}{1+\frac{x+y}{2\sqrt{xy}}+\frac{y+z}{2\sqrt{yz}}+\frac{x+z}{2\sqrt{xz}}}\right] \ .$$

Consider the changes $x \to y \to z$.

(3) $$H_6(\frac{z}{x}) = \frac{z-x}{\sqrt{zx}} = \frac{z-y}{\sqrt{zx}} + \frac{y-x}{\sqrt{zx}}$$

$$= \frac{\sqrt{zy}}{\sqrt{zx}} H_6(\frac{z}{y}) + \frac{\sqrt{xy}}{\sqrt{zx}} H_6(\frac{y}{x})$$

$$= \sqrt{\frac{y}{z}}\, H_6(\frac{y}{x}) + \sqrt{\frac{y}{x}}\, H_6(\frac{z}{y}) \ .$$

This corresponds to equations (15) and (16) for H_1 and H_3 respectively, but here we continue directly from the first row.

(4) $$H_6(\frac{z}{x}) = H_6(\frac{y}{x}) + H_6(\frac{z}{y}) + \left[\frac{z-y}{\sqrt{zx}} - \frac{z-y}{\sqrt{zy}} + \frac{y-x}{\sqrt{zx}} - \frac{y-x}{\sqrt{yx}}\right]$$

$$\epsilon = (z-y)(\frac{\sqrt{y}-\sqrt{x}}{\sqrt{xyz}}) + (y-x)(\frac{\sqrt{y}-\sqrt{z}}{\sqrt{xyz}})$$

$$= \frac{1}{\sqrt{xyz}}\left[\frac{(z-y)(y-x)}{\sqrt{x}+\sqrt{y}} + \frac{(y-x)(y-z)}{\sqrt{y}+\sqrt{z}}\right]$$

$$= \frac{(y-x)(z-y)}{\sqrt{xyz}}\left[\frac{1}{\sqrt{x}+\sqrt{y}} - \frac{1}{\sqrt{y}+\sqrt{z}}\right]$$

$$= (\frac{y-x}{\sqrt{xy}})(\frac{z-y}{\sqrt{z}})\left[\frac{\sqrt{z}-\sqrt{x}}{(\sqrt{x}+\sqrt{y})(\sqrt{y}+\sqrt{z})}\right]$$

$$= (\frac{y-x}{\sqrt{xy}})(\frac{z-y}{\sqrt{z}})(z-x)\frac{1}{(\sqrt{x}+\sqrt{y})(\sqrt{y}+\sqrt{z})(\sqrt{z}+\sqrt{x})}$$

$$= (\frac{y-x}{\sqrt{xy}})(\frac{z-y}{\sqrt{yz}})(\frac{z-x}{\sqrt{zx}})\left[\frac{1}{2+\frac{x+y}{\sqrt{xy}}+\frac{y+z}{\sqrt{yz}}+\frac{x+z}{\sqrt{xz}}}\right] ,$$

from which (2) can already be seen.

Appendix 2. Functional equation $H(xy) = H(x) + H(y)$.

Theorem. The only differentiable solutions $H: \mathbb{R}_+ \to \mathbb{R}$ of the functional equation

$$(1) \qquad \forall x \in \mathbb{R}_+ : \forall y \in \mathbb{R}_+ : H(xy) = H(x) + H(y)$$

are of the form $H(x) = c\log_e x$, where c is an arbitrary real constant.

Proof. Set $y = 1$ to get $H(x \cdot 1) = H(x) + H(1)$, which shows that $H(1) = 0$.

We have for all $x>0$ and $y>0$

$$(2) \qquad H(xy) - H(x) = H(y) - H(1) .$$

Dividing by $xy-x$, when $y \neq 1$, gives

$$(3) \qquad \frac{H(xy) - H(x)}{xy - x} = \frac{H(y) - H(1)}{xy - x} = \frac{1}{x}\left(\frac{H(y) - H(1)}{y - 1}\right).$$

Using the definition of a derivative and the differentiability assumption we get, when $y \to 1$,

$$(4) \qquad H'(x) = \frac{1}{x}H'(1) .$$

This implies that H is continuously differentiable.

The general solution of the differential equation $y' = c/x$ is $H(x) = c\log_e x + d$, where c and d are arbitrary real constants.

Here we must have $d = 0$, because $H(1) = 0$.

$H(x) = c\log_e x$ really is a solution and our theorem is proved.

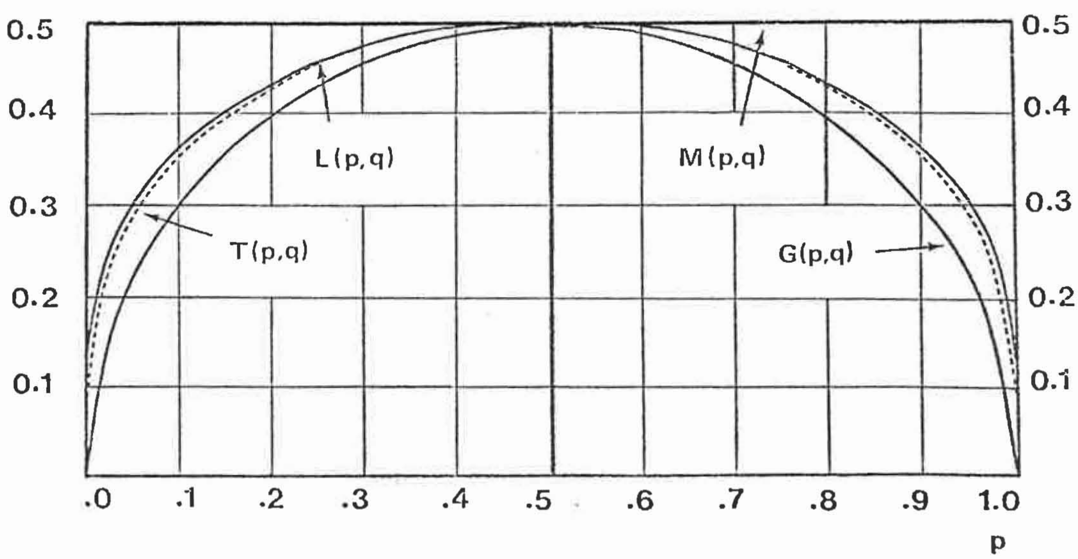
The case $c = 0$ is the trivial solution. □

Appendix 3. On means

Let $K(x,y)$ be some mean of x and y. From the property $K(ax,ay) = aK(x,y)$ we have $K(x,y) = (x+y)K(p,q)$, where $p = x/(x+y)$ and $q = y(x+y)$, and hence, $p+q = 1$. In consequence, we may consider $K(p,q)$, when $p+q = 1$, which determines the mean in question for any numbers x and y. In particular, $K(x,y)/\frac{1}{2}(x+y) = 2K(p,q)$.

Consider the following means $K(x,y)$: $M(x,y) = (x+y)/2$, $G(x,y) = \sqrt{xy}$, $T(x,y) = \sqrt[3]{xy(x+y)/2}$, $L(x,y) = (x-y)/\log(x/y)$. The functions $K(p,q)$ for these means are respectively $1/2$, $\sqrt{pq}$, $\sqrt[3]{pq/2}$, $(p-q)/\log(p/q)$. These are represented graphically in the following figure.

Figure 1. Some means $K(p,q)$



Except for the points $p = 0$, $p = \frac{1}{2}$ and $p = 1$, the following inequalities hold true:

$$0 < \sqrt{pq} < \sqrt{pq}/2 < \frac{p-q}{\log(p/q)} < \frac{1}{2} .$$

These means are used in the index number formulas based on log-changes, suggested by Walsh in 1901, Theil in 1973, Vartia in 1974 and Törnqvist in 1936. Means are usually applied in the vicinity of the point $p = \frac{1}{2}$, in which case all the means are approximately equal to one-half.

Appendix 4. The Sum of the Weights in the Vartia Index I

We shall show that the sum of the weights of the Vartia Index I,

$$(1) \qquad w_i = \frac{L(v_i^1, v_i^0)}{L(V^1, V^0)}$$

is less than or equal to one and that it will be equal to one only if $w_i^1 = w_i^0$ for all the commodities a_i.

The inequality $\Sigma w_i \leq 1$ may be written as

$$(2) \qquad \sum_{i=1}^{n} L(v_{1i}, v_{0i}) \leq L(\sum_{i=1}^{n} v_{1i}, \sum_{i=1}^{n} v_{0i}),$$

where the notation has been simplified to some extent. This can be demonstrated by making use of the properties of the mean of concave functions as follows.

Consider the right-hand side of (2):

$$(3) \qquad L(\sum_{i=1}^{n} v_{1i}, \sum_{i=1}^{n} v_{0i}) = (\Sigma v_{0i})\, L(\frac{\Sigma v_{1i}}{\Sigma v_{0i}}, 1)$$

$$= (\Sigma v_{0i})\, L(\sum_{i=1}^{n} w_{0i} \frac{v_{1i}}{v_{0i}}, 1),$$

where $w_{0i} = v_{0i}/\Sigma v_{0i}$. $L(s,1) = (s-1)/\log s$ is a concave function, and thus, by virtue of the so-called Jenssen inequality[1),]

---

1) See, e.g., Saaty (1959) p. 114.

$$(4) \qquad (\Sigma v_{0i}) L(\sum_{i=1}^{n} w_{0i} \frac{v_{1i}}{v_{0i}}, 1) \geq (\sum_{1}^{n} v_{0i}) \sum_{i=1}^{n} w_{0i} L(\frac{v_{1i}}{v_{0i}}, 1)$$

$$= 1 \cdot \sum_{i=1}^{n} L(v_{1i}, v_{0i}).$$

Here, as in (3), use has been made of the property $aL(x,y) = L(ax,ay)$, and (2) consequently follows from this.

In an entirely corresponding manner it can be shown that for Theil's mean $T(x,y) = \sqrt[3]{xy(\frac{x+y}{2})}$, for example, we have

$$(5) \qquad \sum_{i=1}^{n} T(v_{1i}, v_{0i}) \leq T(\sum_{i=1}^{n} v_{1i}, \sum_{i=1}^{n} v_{0i}).$$

A corresponding equation holds good generally for any mean $K(x,y)$ provided that $K(s,1)$ is a concave function of s, in which case the sum of the weights,

$$(6) \qquad w_i = \frac{K(v_{1i}, v_{0i})}{K(\Sigma v_{1i}, \Sigma v_{0i})}$$

is consequently one at most. In the special case $K(x,y) = M(x,y) = \frac{1}{2}(x+y)$ the sum of the weights is identically equal to unity.

The equality sign in (4) applies only if the terms $L(\frac{v_{1i}}{v_{0i}}), 1)$ are all equal, in which case we have, for all values of i,

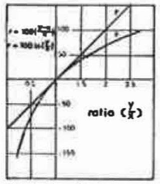
$$(7) \qquad L(\frac{\Sigma v_{1i}}{\Sigma v_{0i}}, 1) = L(\frac{v_{1i}}{v_{0i}}, 1)$$

or v_{1i}/v_{0i} is a constant. The values of all commodities a_i will then change in the same proportion and, hence, $w_{1i} = w_{0i}$, as was asserted. In this case the weights (6) are independent of the choice of the mean $K(x,y)$ and will coincide with the value shares $w_{0i} = w_{1i}$.

This result shows that the weighting (6) will behave completely reasonably in situations where, owing to a rapid rate of inflation, the values of all commodities rise in proportion, or $v_{1i} = kv_{0i}$, $k>>1$.

APPENDIX 5. Percentages and Dyns[1]

The correspondence between two indicators of relative changes



Percentage change from x to y: $r = 100(\frac{y-x}{x})$

Dynamic change from x to y: $\dot{r} = 100 \ln(\frac{y}{x})$

The transformation $\dot{r} = 100 \ln(1 + \frac{r}{100})$ gives percentage changes r in terms of dynamic changes $\dot{r}$. For small changes $\dot{r} \approx r$, or, in other words, the percentages and dyns correspond approximately to each other.

r per cent	ṙ dyn	r per cent	ṙ dyn	r per cent	ṙ dyn	r per cent	ṙ dyn
-100	-∞	-60	-91.629	-20.0	-22.314	0.0	0:000
-99	-460.517	-59	-89.160	-19.5	-21.691	0.5	0.499
-98	-391.202	-58	-86.750	-19.0	-21.072	1:0	0.995
-97	-350.656	-57	-84.397	-18.5	-20.457	1.5	1.489
-96	-321.888	-56	-82.098	-18.0	-19.845	2.0	1.980
-95	-299.573	-55	-79.851	-17.5	-19.237	2.5	2.469
-94	-281.341	-54	-77.653	-17.0	-18.633	3.0	2.956
-93	-265.926	-53	-75.502	-16.5	-18.032	3.5	3.440
-92	-252.573	-52	-73.397	-16.0	-17.435	4.0	3.922
-91	-240.795	-51	-71.335	-15.5	-16.842	4.5	4.402
-90	-230.259	-50	-69.315	-15.0	-16.252	5.0	4.879
-89	-220.727	-49	-67.334	-14.5	-15.665	5.5	5.354
-88	-212.026	-48	-65.393	-14.0	-15.082	6.0	5.827
-87	-204.022	-47	-63.488	-13.5	-14.503	6.5	6.297
-86	-196.611	-46	-61.619	-13.0	-13.926	7.0	6.766
-85	-189.712	-45	-59.784	-12.5	-13.353	7.5	7.232
-84	-183.258	-44	-57.982	-12.0	-12.783	8.0	7.696
-83	-177.196	-43	-56.212	-11.5	-12.217	8.5	8.158
-82	-171.480	-42	-54.473	-11.0	-11.653	9.0	8.618
-81	-166.073	-41	-52.763	-10.5	-11.093	9.5	9.075
-80	-160.944	-40	-51.083	-10.0	-10.536	10.0	9.531
-79	-156.065	-39	-49.430	-9.5	-9.982	10.5	9.985
-78	-151.413	-38	-47.804	-9.0	-9.431	11.0	10.436
-77	-146.968	-37	-46.204	-8.5	-8.883	11.5	10.885
-76	-142.712	-36	-44.629	-8.0	-8.338	12.0	11.333
-75	-138.629	-35	-43.078	-7.5	-7.796	12.5	11.778
-74	-134.707	-34	-41.552	-7.0	-7.257	13.0	12.222
-73	-130.933	-33	-40.048	-6.5	-6.721	13.5	12.663
-72	-127.297	-32	-38.566	-6.0	-6.188	14.0	13.103
-71	-123.787	-31	-37.106	-5.5	-5.657	14.5	13.540
-70	-120.397	-30	-35.667	-5.0	-5.129	15.0	13.976
-69	-117.118	-29	-34.249	-4.5	-4.604	15.5	14.410
-68	-113.943	-28	-32.850	-4.0	-4.082	16.0	14.842
-67	-110.866	-27	-31.471	-3.5	-3.563	16.5	15.272
-66	-107.881	-26	-30.111	-3.0	-3.046	17.0	15.700
-65	-104.982	-25	-28.768	-2.5	-2.532	17.5	16.127
-64	-102.165	-24	-27.444	-2.0	-2.020	18.0	16.551
-63	-99.425	-23	-26.136	-1.5	-1.511	18.5	16.974
-62	-96.758	-22	-24.846	-1.0	-1.005	19.0	17.395
-61	-94.161	-21	-23.572	-0.5	-0.501	19.5	17.815

1) Translated from Herva & Vartia & Vasama (1973)

r per cent	ṙ dyn	r per cent	ṙ dyn	r per cent	ṙ dyn	r per cent	ṙ dyn
20	18.232	60	47.000	100	69.315	500	179.176
21	19.062	61	47.623	110	74.194	510	180.829
22	19.885	62	48.243	120	78.846	520	182.455
23	20.701	63	48.858	130	83.291	530	184.055
24	21.511	64	49.470	140	87.547	540	185.630
25	22.314	65	50.078	150	91.629	550	187.180
26	23.111	66	50.682	160	95.551	560	168.707
27	23.902	67	51.282	170	99.325	570	190.211
28	24.686	68	51.879	180	102.962	580	191.692
29	25.464	69	52.473	190	106.471	590	193.152
30	26.236	70	53.063	200	109.861	600	194.591
31	27.003	71	53.649	210	113.140	610	196.009
32	27.763	72	54.232	220	116.315	620	197.408
33	28.518	73	54.812	230	119.392	630	198.787
34	29.267	74	55.389	240	122.378	640	200.148
35	30.010	75	55.962	250	125.276	650	201.490
36	30.748	76	56.531	260	128.093	660	202.815
37	31.481	77	57.098	270	130.833	670	204.122
38	32.208	78	57.661	280	133.500	680	205.412
39	32.930	79	58.222	290	136.098	690	206.686
40	33.647	80	58.779	300	138.629	700	207.944
41	34.359	81	59.333	310	141.099	710	209.186
42	35.066	82	59.884	320	143.508	720	210.413
43	35.767	83	60.432	330	145.861	730	211.626
44	36.464	84	60.977	340	148.160	740	212.823
45	37.156	85	61.519	350	150.408	750	214.007
46	37.844	86	62.058	360	152.606	760	215.176
47	38.526	87	62.594	370	154.756	770	216.332
48	39.204	88	63.127	380	156.862	780	217.475
49	39.878	89	63.658	390	158.924	790	218.605
50	40.547	90	64.185	400	160.944	800	219.722
51	41.211	91	64.710	410	162.924	810	220.827
52	41.871	92	65.233	420	164.866	820	221.920
53	42.527	93	65.752	430	166.771	830	223.001
54	43.178	94	66.269	440	168.640	840	224.071
55	43.825	95	66.783	450	170.475	850	225.129
56	44.469	96	67.294	460	172.277	860	226.176
57	45.108	97	67.803	470	174.047	870	227.213
58	45.742	98	68.310	480	175.786	880	228.238
59	46.373	99	68.813	490	177.495	890	229.253

EXAMPLES:

1. A decrease of 10 per cent, $r = -10$, corresponds to -10.536 dyn ($\dot{r} = -10.536$). An increase of 10 per cent, $r = +10$, corresponds to 9.531 dyn ($\dot{r} = 9.351$).
2. A decrease to a half ($r = -50$ %) corresponds to a decrease of 69.351 dyn ($\dot{r} = -69.315$). An increase to the double of the original value ($r = +100$ %) corresponds to an increase of 69.315 dyn ($\dot{r} = +69.315$). In this sense the dynamic change measures relative increases and decreases symmetrically.
3. Compound interest: growth at an annual rate of 5 per cent for 20 years is computed by means of the dynamic changes as follows:

   5 per cent = 4.879 dyn
   20 x 4.879 dyn = 97.58 dyn
   97.58 dyn = 165 per cent

   Thus, if interest is compounded at an annual rate of 5 per cent for 20 years, an amount of 100 marks will increase by 165 marks to 265 marks.

Appendix 6. Variances and covariances of value, price and quantity log-changes

Consider price and quantity indices P_0^1 and Q_0^1 for which

(1) $\log P_0^1 = \Sigma w_i \log(p_i^1/p_i^0)$

(2) $\log Q_0^1 = \Sigma w_i \log(q_i^1/q_i^0)$,

where w_i are some nonnegative weights the sum of which may or may not equal unity. Define logarithmic price and quantity deviations from their means (1) and (2) by

(3) $\dot{p}_i = \log(p_i^1/p_i^0) - \log P_0^1$

(4) $\dot{q}_i = \log(q_i^1/q_i^0) - \log Q_0^1$.

Then we have by (1) and (2)

(5) $\Sigma w_i \dot{p}_i = 0$

(6) $\Sigma w_i \dot{q}_i = 0$.

Define the variances of price and quantity log-changes by

(7) $s_p^2 = \Sigma w_i \dot{p}_i^2 = \Sigma w_i [\log(p_i^1/p_i^0)]^2 - [\log P_0^1]^2$

(8) $s_q^2 = \Sigma w_i \dot{q}_i^2 = \Sigma w_i [\log(q_i^1/q_i^0)]^2 - [\log Q_0^1]^2$

and the covariance of price and quantity log-changes by

(9) $\mathrm{cov}(\dot{p},\dot{q}) = \Sigma w_i \dot{p}_i \dot{q}_i = \Sigma w_i \log(p_i^1/p_i^0) \log(q_i^1/q_i^0) - \log P_0^1 \log Q_0^1$

If $\Sigma w_i = 1$ and w_i is considered to be the probability of the two dimensional discrete random variable $(\log(p_i^1/p_i^0, \log(q_i^1/q_i^0))$, then (1) and (2) are the expectations (means) of the corresponding variables and (7), (8) and (9) are literally their variances and covariance, see Törnqvist (1937) , Rajaoja (1957) p. 56 and Theil (1967) p. 154. We also use the notation

(10) $$\text{cov}(\dot{p},\dot{p}) = s_p^2$$

(11) $$\text{cov}(\dot{q},\dot{q}) = s_q^2 \quad .$$

Consider also the mean of log-changes in values defined as (1) and (2)

(12) $$\log V_0^1 = \Sigma w_i \log(v_i^1/v_i^0) = \log P_0^1 + \log Q_0^1 \quad .$$

Usually $\log V_0^1$ is not equal to the log-change in the total value, $\log(V^1/V^0)$. However, if the index formulas (1) and (2) satisfy the factor reversal test we have

(13) $$\log V_0^1 = \log P_0^1 + \log Q_0^1 = \log(V^1/V^0) \quad .$$

This applies if we use as weights w_i the weights of Vartia Index I or II. Define the logarithmic value deviation from its mean (12) by

(14) $$\dot{v}_i = \log(v_i^1/v_i^0) - \log V_0^1 = \dot{p}_i + \dot{q}_i$$

and the variance of $\dot{v}_i$ or $\log(v_i^1/v_i^0)$ by

(15) $$s_v^2 = \text{cov}(\dot{v},\dot{v}) = \Sigma w_i \dot{v}_i^2 \quad .$$

Then we have identically

(17) $$s_v^2 = \Sigma w_i (\dot{p}_i^2 + 2\dot{p}_i \dot{q}_i + \dot{q}_i^2)$$
$$= s_p^2 + 2\text{cov}(\dot{p},\dot{q}) + s_q^2 \quad .$$

Theil (1967) p. 154 and Theil (1973) considered instead of (14) the logarithmic deviations of the value shares from their mean

$$(18) \quad \dot{w}_i = \log(w_i^1/w_i^0) - \Sigma w_i \log(w_i^1/w_i^0)$$

$$= \log\left(\frac{v_i^1/v_i^0}{V^1/V^0}\right) - \Sigma w_i \log\left(\frac{v_i^1/v_i^0}{V^1/V^0}\right)$$

$$= \log(v_i^1/v_i^0) - \Sigma w_i \log(v_i^1/v_i^0) + (\Sigma w_i - 1) \log(V^1/V^0) \quad .$$

But these are identical with $\dot{v}_i$:s if $\Sigma w_i = 1$ as Theil supposes. For the weights w_i of the Vartia Index I or II we have (13) and hence (14) changes into

$$(19) \quad \dot{v}_i = \log(v_i^1/v_i^0) - \log(V^1/V^0)$$

$$= \log(w_i^1/w_i^0) \quad .$$

Thus for the weights of Vartia Indices I and II, (17) changes to

$$(20) \quad s_v^2 = \Sigma w_i \, [\log(v_i^1/v_i^0) - \log(V^1/V^0)]^2$$

$$= \Sigma w_i [\log(w_i^1/w_i^0)]^2 = s_p^2 + 2\mathrm{cov}(\dot{p},\dot{q}) + s_q^2$$

Because for the weights of Vartia Index II we have, in addition, Theil's condition

$$(21) \quad \Sigma w_i \log(w_i^1/w_i^0) = 0 \text{ and } \Sigma w_i = 1$$

we have for them also

$$(22) \quad \Sigma w_i \dot{w}_i^2 = \Sigma w_i [\log(v_i^1/v_i^0) - \log(V^1/V^0)]^2 \quad .$$

In our table 3 of chapter 7 we have presented the variances of price, quantity and value log-changes, i.e., (7), (8) and (15), and the covariance (9) as calculated using the weights of the Vartia Index I.

Appendix 7. The GDP-data: revision of published figures

4. Eri elinkeinojen tuotannon volyymi-indeksit [1]) vuosina 1964–1972
Index för produktionsvolymen [1]) enligt näringsgrenar åren 1964–1972
Indexes [1]) of production for different industries in 1964–1972
1964 = 100

	1964	1965	1966	1967	1968	1969	1970	1971	1972*
Maatalous, metsätalous, metsästys ja kalastus – Lantbruk, skogsbruk, jakt och fiske – *Agriculture, forestry, hunting and fishing*	100	98.20	92.30	92.60	95.46	99.54	102.96	101.64	95.10
Maatalous – Lantbruk – *Agriculture*	100	94.25	96.31	95.48	96.87	96.69	94.65	96.82	93.89
Metsätalous – Skogsbruk – *Forestry*	100	102.08	87.29	88.59	92.51	101.81	110.73	105.89	95.25
Metsästys ja kalastus – Jakt och fiske – *Hunting and fishing*	100	108.19	108.59	114.36	130.60	119.73	134.09	125.91	123.89
Kaivannaisteollisuus – Gruv- o.a. extraktiv industri – *Mining and quarrying*	100	106.20	97.11	107.23	108.20	121.23	130.65	104.75	118.73
Tehdasteollisuus – Fabriksindustri – *Manufacturing*	100	106.24	111.65	115.31	121.14	136.86	153.27	157.19	175.77
Elintarviketeollisuus – Livsmedelsindustri – *Food manufacturing industries, except beverage industries*	100	106	113	118	120	127	133	138	146
Juomia valmistava teollisuus – Dryckesvaruindustri – *Beverage industries*	100	112	124	136	150	208	234	231	254
Tupakkateollisuus – Tobaksindustri – *Tobacco manufactures*	100	110	112	128	128	129	132	139	160
Tekstiiliteollisuus – Textilindustri – *Manufacture of textiles*	100	96	105	108	110	129	135	136	146
Kenkä-, vaatetus- ja ompeluteollisuus – Sko-, konfektions- och sömnadsindustri — *Manufacture of footwear, other wearing apparel and made-up textile goods*	100	98	109	114	110	128	146	148	169
Puuteollisuus – Träindustri – *Manufacture of wood and cork, except manufacture of furniture*	100	103	103	106	112	127	137	140	139
Huonekalu- ja rakennuspuusepänteollisuus – Möbel- och byggnadssnickeriindustri – *Manufacture of furniture and fixtures, except manufacture of metal furniture*	100	107	111	114	113	135	148	161	190
Paperiteollisuus – Pappersindustri – *Manufacture of paper and paper products*	100	107	112	110	118	131	137	138	151
Graafinen teollisuus – Grafisk industri – *Printing, publishing and allied industries*	100	102	107	105	112	117	126	129	135
Nahka- ja nahkateosteollisuus – Skinn-, läder- och lädervaruindustri – *Manufacture of leather and leather products, except footwear*	100	100	108	113	118	133	142	141	153
Kumiteollisuus – Gummiindustri – *Manufacture of rubber products*	100	110	131	131	133	159	175	174	163
Kemian teollisuus – Kemisk industri – *Manufacture of chemicals and chemical products*	100	111	117	127	138	155	183	203	238
Kiviöljy- ja asfalttiteollisuus – Mineralolje- och asfaltindustri – *Manufacture of products of petroleum and asphalt*	100	112	143	164	206	274	313	334	372
Savi-, lasi- ja kivenjalostusteollisuus – Ler-, glas- och stenförädlingsindustri – *Manufacture of non-metallic mineral products, except products of petroleum and coal*	100	111	121	124	132	154	186	195	209
Metallien perusteollisuus – Metallverk – *Basic metal industries*	100	120	118	119	134	161	175	155	203
Metallituoteteollisuus – Metallmanufaktur – *Manufacture of metal products, except machinery and transport equipment*	100	107	116	119	123	141	165	161	186
Koneteollisuus – Maskinindustri – *Manufacture of machinery except electrical machinery*	100	107	109	111	116	130	156	170	200
Sähköteknillinen teollisuus – Elektroteknisk industri – *Manufacture of electrical machinery, apparatus, appliances and supplies*	100	103	104	109	116	137	181	193	225

[1]) Laskettu vuoden 1964 hintaisten bruttokansantuote-sarjojen (SNA) perusteella. – Beräknats på basen av inhemsk bruttoprodukt (SNA) serier till 1964 års priser. –*Calculated on basis of gross domestic product (SNA) series at 1964 prices.*

Appendix 7. (continued)

	1964	1965	1966	1967	1968	1969	1970	1971	1972*
Kulkuneuvoteollisuus – Transportmedelsindustri – *Manufacture of transport equipment*	100	108	109	114	120	125	137	126	146
Muu tehdasteollisuus – Annan fabriksindustri – *Miscellaneous manufacturing industries*	100	112	116	142	148	201	239	250	307
Rakennustoiminta – Byggnadsverksamhet – *Construction*	100	108.93	110.47	112.68	108.82	116.40	125.53	124.32	132.82
Talonrakennustoiminta – Husbyggnadsverksamhet – *House construction*	100	112.20	112.78	116.68	109.42	123.19	140.69	138.43	149.40
Maa- ja vesirakennustoiminta – Anläggningsverksamhet – *Other construction*	100	103.10	106.34	105.56	107.74	104.29	98.48	99.13	103.23
Sähkö-, kaasu-, vesijohto- yms. laitokset – El-, gas- och vattenverk m.m. – *Electricity, gas, water and sanitary services*	100	106.87	117.48	121.98	128.15	149.62	165.52	172.49	191.62
Liikenne – Samfärdsel – *Transport and communication*	100	105.11	108.89	108.81	113.80	123.60	132.41	134.09	144.49
Vesiliikenne – Sjöfart – *Water transport*	100	101	103	104	115	128	140	137	149
Rautatieliikenne – Järnvägstrafik – *Railway transport*	100	106	112	112	110	114	117	113	126
Tieliikenne – Vägtrafik – *Road transport*	100	103	102	97	98	105	112	113	116
Tietoliikenne – Post, telefon, telegraf m.m. – *Communications*	100	110	121	128	140	153	166	175	190
Muu – Övrig – *Other*	100	113	118	122	126	147	159	175	197
Kauppa – Handel – *Commerce*	100	107.16	109.71	111.66	108.66	121.99	132.30	138.15	151.33
Tukkukauppa – Partihandel – *Wholesale trade*	100	105.15	108.61	112.78	108.17	127.99	142.90	150.23	165.38
Vähittäiskauppa – Detaljhandel – *Retail trade*	100	108.66	110.54	110.83	109.03	117.49	124.37	129.09	140.81
Pankit ja vakuutuslaitokset – Banker och försäkringsanstalter – *Banking and insurance*	100	106.07	110.63	119.78	121.22	126.35	133.03	150.13	166.37
Pankit – Banker – *Banking*	100	106.59	111.39	120.24	124.37	129.13	139.76	160.03	181.59
Vakuutuslaitokset – Försäkringsanstalter – *Insurance*	100	104.90	108.96	118.75	114.24	122.10	118.11	128.16	132.60
Asuntojen omistus – Bostäder – *Ownership of dwellings*	100	104.20	108.76	113.76	118.44	123.33	129.14	135.23	142.17
Yleinen hallinto ja maanpuolustus – Offentlig förvaltning och landsförsvar – *Public administration and defence*	100	103.77	110.48	113.88	118.82	123.41	128.15	134.58	139.78
Palvelukset – Tjänster – *Services*	100	105.45	108.67	113.57	119.24	126.72	133.69	140.07	148.16
Opetus – Undervisning – *Education*	100	104.20	106.56	110.88	116.13	121.35	125.63	130.78	137.63
Terveydenhoito – Hälsovård – *Medical and health services*	100	107.72	110.89	118.75	126.80	136.76	145.02	152.59	162.77
Virkistys ja huvittelu – Rekreation och nöjen – *Recreation and entertainment*	100	100.80	103.28	107.99	114.20	122.72	129.72	138.51	147.47
Ravitsemus- ja majoitusliikkeet – Förplägnads- och härbärgeringsverksamhet – *Catering trade*	100	105.21	109.64	113.04	121.27	131.39	146.67	160.97	176.93
Henkilökohtaiset palvelukset – Personliga tjänster – *Personal services*	100	101.04	103.22	105.56	107.69	111.74	116.31	120.72	124.99
Muut – Övriga – *Other*	100	108.39	113.67	117.70	121.31	130.45	138.77	142.82	148.58
Bruttokansantuote – Inhemsk bruttoprodukt – *Gross domestic product*	100.0	104.7	107.0	109.9	113.0	122.9	132.4	135.6	144.8
Alkutuotanto [2]) – Primärproduktion [2]) – *Primary production [2])*	100	98	92	93	96	100	104	102	96
Jalostuselinkeinot [3]) – Sekundär produktion [3]) – *Secondary production [3])*	100	107	112	115	119	133	173	151	165
Palveluelinkeinot [4]) – Tjänster [4]) – *Services [4])*	100	105	109	113	116	124	110	138	148

[2]) Alkutuotantoa ovat maa- ja metsätalous, metsästys ja kalastus sekä kaivannaisteollisuus. – Till primär produktion räknas lant- och skogsbruk, jakt och fiske samt gruv- o.a. extraktiv industri. – *Primary production includes agriculture, forestry, hunting and fishing, mining and quarrying.*

[3]) Jalostuselinkeinoja ovat tehdasteollisuus, rakennustoiminta ja sähkö-, kaasu-, vesijohto- yms. laitokset. – Till sekundär produktion räknas fabriksindustri, byggnadsverksamhet samt el-, gas- och vattenverk m.m. – *Secondary production includes industry and construction.*

[4]) Palveluelinkeinoja ovat liikenne, kauppa, pankit, vakuutus, asuntojen omistus, yleinen hallinto ja maanpuolustus sekä palvelukset. – Till tjänster räknas samfärdsel, handel, banker, försäkring, bostäder, offentlig förvaltning och landsförsvar samt tjänster. – *Services include transport and communication, commerce, ownership of dwellings, public administration and defence and services.*

Appendix 8. The imports data

Imports of fuels and lubricant were divided according to SITC (The Standard International Trade Classification) into the following 8 groups and subgroups:

Set notation	SITC no	Explanation
A_1	321.4	Coal (anthracite, bituminous)
A_2	321.8	Coke and semi-coke of coal, of lignite or of peat
A_3	331	Petroleum, crude and partly refined for further refining (excluding natural gasolene)
A_4	332.1	Motor spirit (gasolene and other light oils for similar uses, including natural gasolene)
A_5	332.2	Lamp oil and white spirit (kerosene, illuminating oil, jet fuel)
A_6	332.3	Distillate fuels
A_7	332.4	Residual fuel oils
A_8	332.5	Lubricating oils and greases (including mixtures with animal and vegatable lubricants)
A		Fuels and lubricants

The subgroups 321.5 (Briquettes of coal) and 321.6 (Lignite briquettes and lignite) were ignored because of their very small imports.

In the following tables we give the monthly imports of $A_1,\ldots,A_8$ as calculated from the cumulative sums in the monthly bulletins published by the Board of Customs (1972, 1973), table 2.

Commodity group 1:

SITC 321.4, Coal

period		quantity mill.kg	value mill.mk	price p/kg
1972	1	92.565	8.093	8.743
	2	38.681	3.057	7.903
	3	69.122	4.485	6.489
	4	47.107	2.782	5.906
	5	162.778	9.283	5.703
	6	240.033	13.214	5.505
	7	325.462	17.667	5.428
	8	316.580	17.632	5.570
	9	309.031	17.580	5.689
	10	384.218	21.170	5.510
	11	334.683	18.393	5.496
	12	342.580	19.582	5.716
1973	1	203.410	11.528	5.667
	2	144.574	8.648	5.982
	3	181.161	9.566	5.280
	4	181.134	9.686	5.347
	5	166.916	8.962	5.369
	6	124.219	7.320	5.893
	7	139.522	7.888	5.654
	8	381.552	19.649	5.150
	9	353.191	19.565	5.539
	10	372.774	20.317	5.450
	11	428.530	23.696	5.530
	12	289.884	17.165	5.921
1974	1	356.099	23.486	6.595
	2	254.445	26.948	10.591
	3	179.999	19.368	10.760
	4	227.273	25.993	11.437
	5	294.404	38.166	12.964
	6	267.411	34.368	12.852
	7	402.773	51.915	12.889
	8	499.982	64.972	12.995
	9	421.573	56.472	13.396
	10			
	11			
	12			

Commodity group 2:

SITC 321.8, Coke and semi-coke of coal

period		quantity	value	price
		mill.kg	mill.mk	p/kg
1972	1	85.573	18.401	21.503
	2	41.108	7.167	17.435
	3	14.701	2.077	14.128
	4	46.499	7.554	16.246
	5	64.867	10.599	16.340
	6	50.126	7.245	14.454
	7	75.344	11.068	14.690
	8	66.606	9.826	14.752
	9	89.347	14.647	16.393
	10	68.508	11.832	17.271
	11	40.121	6.527	16.268
	12	79.010	12.330	15.606
1973	1	84.439	13.696	16.220
	2	42.212	6.995	16.571
	3	11.135	.850	7.634
	4	67.359	11.437	16.979
	5	84.666	14.197	16.768
	6	27.676	4.092	14.785
	7	103.795	16.939	16.320
	8	88.295	14.638	16.579
	9	46.807	7.146	15.267
	10	119.092	20.236	16.992
	11	85.165	14.995	17.607
	12	71.003	11.523	16.229
1974	1	97.019	16.559	17.068
	2	48.382	9.052	18.709
	3	55.819	10.527	18.859
	4	74.985	13.530	18.044
	5	64.013	12.390	19.355
	6	96.532	17.937	18.581
	7	83.582	16.315	19.520
	8	144.304	29.750	20.616
	9	92.509	21.459	23.197
	10			
	11			
	12			

Commodity group 3:

SITC 331, Petroleum, crude and partly refined for further refining

period	quantity mill.kg	value mill.mk	price p/kg
1972 1	781.353	69.017	8.833
2	485.193	45.383	9.354
3	423.789	39.352	9.286
4	567.939	54.866	9.661
5	683.281	63.148	9.242
6	1024.405	92.132	8.994
7	1227.846	105.850	8.621
8	739.857	62.263	8.416
9	767.634	67.479	8.791
10	927.712	81.867	8.825
11	943.982	81.673	8.652
12	661.489	59.401	8.980
1973 1	999.381	97.493	9.755
2	352.986	33.115	9.381
3	556.759	54.182	9.732
4	836.834	79.706	9.525
5	531.688	50.885	9.570
6	740.563	71.377	9.638
7	970.194	92.622	9.547
8	951.934	87.667	9.209
9	777.014	75.621	9.732
10	887.207	94.370	10.637
11	1256.365	190.297	15.147
12	661.109	118.688	17.953
1974 1	991.175	227.633	22.966
2	755.586	245.176	32.448
3	464.018	169.200	36.464
4	740.365	261.724	35.351
5	612.093	204.872	33.471
6	876.905	314.820	35.901
7	909.525	307.497	33.809
8	1005.109	325.919	32.426
9	627.050	215.469	34.362
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Commodity group 4:

SITC 332.1, Motor spirit, etc.

period	quantity	value	price
	mill. l	mill.mk	p/l
1972 1	.600	.100	16.667
2	8.284	.804	9.705
3	.428	.093	21.729
4	.244	.058	23.770
5	.934	.169	18.094
6	8.263	.840	10.166
7	6.978	.528	7.567
8	3.509	.758	21.602
9	7.851	.727	9.260
10	32.594	4.636	14.223
11	.021	.016	76.190
12	13.548	1.028	7.588
1973 1	5.793	.781	13.482
2	8.542	.935	10.946
3	.341	.086	25.220
4	.315	.071	22.540
5	.078	.039	50.000
6	4.023	.903	22.446
7	18.833	4.065	21.584
8	7.265	1.492	20.537
9	.320	.088	27.500
10	20.853	3.520	16.880
11	85.487	14.867	17.391
12	23.264	3.950	16.979
1974 1	62.273	11.127	17.868
2	.022	.015	68.182
3	11.014	2.000	18.159
4	5.217	2.047	39.237
5	5.504	2.314	42.042
6	4.039	1.879	46.521
7	14.357	5.990	41.722
8	78.374	24.745	31.573
9	4.940	1.880	38.057
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Commodity group 5:

SITC 332.2, Lamp oil and white spirit, etc.

period	quantity	value	price
	mill. l	mill.mk	p/l
1972 1	7.505	.988	13.165
2	.839	.163	19.428
3	5.822	.764	13.123
4	.426	.066	15.493
5	3.283	.489	14.895
6	3.987	.553	13.870
7	1.931	.332	17.193
8	.684	.108	15.789
9	2.991	.441	14.744
10	5.544	.732	13.203
11	.981	.154	15.698
12	4.370	.583	13.341
1973 1	3.602	.564	15.658
2	2.956	.427	14.445
3	2.424	.393	16.213
4	2.698	.410	15.196
5	.885	.139	15.706
6	4.454	.897	20.139
7	1.930	.314	16.269
8	.276	.066	23.913
9	.667	.106	15.892
10	5.305	1.280	24.128
11	.790	.160	20.253
12	1.870	.436	23.316
1974 1	2.527	.787	31.144
2	.979	.398	40.654
3	.869	.362	41.657
4	1.872	.512	27.350
5	5.148	1.979	38.442
6	1.236	1.387	112.217
7	.991	.276	27.851
8	.681	.485	71.219
9	3.088	1.200	38.860
10			
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Commodity group 6:

SITC 332.3, Distillate fuels

period	quantity	value	price
	mill.kg	mill.mk	p/kg
1972 1	168.395	26.972	16.017
2	127.144	20.185	15.876
3	126.876	21.854	17.225
4	163.416	27.021	16.535
5	158.038	25.781	16.313
6	149.870	21.743	14.508
7	234.029	33.404	14.273
8	214.815	32.394	15.080
9	185.758	26.627	14.334
10	202.561	27.694	13.672
11	238.244	34.195	14.353
12	155.572	20.755	13.341
1973 1	158.107	24.652	15.592
2	161.245	25.878	16.049
3	115.921	20.075	17.318
4	74.068	12.962	17.500
5	115.386	19.959	17.298
6	96.564	16.875	17.475
7	149.196	24.812	16.630
8	165.031	28.000	16.967
9	201.530	37.350	18.533
10	179.545	34.449	19.187
11	168.914	32.002	18.946
12	197.617	44.195	22.364
1974 1	240.206	90.362	37.619
2	188.572	96.467	51.157
3	189.263	101.518	53.639
4	171.512	86.498	50.433
5	135.707	55.003	40.531
6	143.549	47.084	32.800
7	132.620	44.072	33.232
8	127.089	44.115	34.712
9	144.677	51.130	35.341
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Commodity group 7:

SITC 332.4, Residual fuel oils

period	quantity mill.kg	value mill.mk	price p/kg
1972 1	69.328	5.615	8.099
2	53.554	4.457	8.322
3	53.890	4.525	8.397
4	59.876	5.553	9.274
5	124.361	10.404	8.366
6	191.849	15.403	8.029
7	156.102	12.031	7.707
8	161.677	12.558	7.767
9	142.755	11.208	7.851
10	151.268	12.095	7.996
11	180.815	13.587	7.514
12	190.181	15.867	8.343
1973 1	108.627	8.633	7.947
2	47.586	4.689	9.854
3	113.714	9.408	8.273
4	42.613	3.620	8.495
5	152.147	12.517	8.227
6	179.523	15.596	8.687
7	223.840	18.353	8.199
8	260.189	19.086	7.335
9	257.772	26.638	10.334
10	326.195	29.586	9.070
11	273.424	25.582	9.356
12	267.842	38.736	14.462
1974 1	223.834	48.156	21.514
2	181.439	70.406	38.804
3	134.265	40.553	30.204
4	82.322	23.206	28.189
5	120.048	30.359	25.289
6	110.836	27.610	24.911
7	143.648	37.176	25.880
8	186.438	46.163	24.761
9	271.479	64.988	23.938
10			
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12			

Commodity group 8:

SITC 332.5, Lubricating oils and greases, etc.

period	quantity mill.kg	value mill.mk	price p/kg
1972 1	12.509	5.411	43.257
2	4.300	2.200	51.163
3	2.727	2.364	86.689
4	11.275	5.409	47.973
5	5.238	2.966	56.625
6	3.875	2.502	64.568
7	9.144	4.514	49.366
8	11.506	5.531	48.071
9	2.599	2.622	100.885
10	8.228	4.191	50.936
11	7.098	4.259	60.003
12	11.451	4.563	39.848
1973 1	9.195	4.528	49.244
2	5.834	3.188	54.645
3	5.484	3.294	60.066
4	5.154	2.825	54.812
5	10.493	4.904	46.736
6	4.198	2.290	54.550
7	5.248	3.138	59.794
8	13.575	6.273	46.210
9	7.568	3.830	50.608
10	9.820	5.488	55.886
11	11.960	5.762	48.177
12	6.414	3.744	58.372
1974 1	19.920	7.843	39.372
2	8.182	5.206	63.627
3	9.729	7.455	76.627
4	11.758	8.948	76.101
5	11.505	9.572	83.199
6	3.017	3.023	100.199
7	10.470	9.128	87.182
8	5.872	5.671	96.577
9	13.473	11.519	85.497
10			
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12			

Appendix 9. Price index formulas P_0^1

Symbols: Price, quantity and value of commodity a_i are p_i, q_i and $v_i = p_i q_i$.

Total value $v = \Sigma v_i$ and value share $w_i = v_i/V$.

Superscripts refer to points (or periods) of time, e.g., p_i^0 and p_i^1 are the old and new price of a_i.

Laspeyres (1864) : $P_1^0(La) = \dfrac{\Sigma p_i^1 q_i^0}{\Sigma p_i^0 q_i^0}$

Paasche (1874) : $P_0^1(Pa) = \dfrac{\Sigma p_i^1 q_i^1}{\Sigma p_i^0 q_i^1}$

Fisher (1911) : $P_0^1 = \sqrt{P_0^1(La) P_0^1(Pa)}$

Stuvel (1957) : $P_0^1 = A + \sqrt{A^2 + \dfrac{V^1}{V^0}}$, where $A = \frac{1}{2}\left(P_0^1(La) - \dfrac{V^1/V^0}{P_0^1(Pa)}\right)$

Price indices based on log-changes

$$\log P_0^1 = \Sigma \bar{w}_i \log\left(\frac{p_i^1}{p_i^0}\right)$$

Walsh (1901) : $\bar{w}_i = \sqrt{w_i^1 w_i^0} \Big/ \sum_j \sqrt{w_j^1 w_j^0} = \dfrac{G(w_i^1, w_i^0)}{\Sigma G(,)}$

"Törnqvist I" : $\bar{w}_i = \dfrac{v_i^1 + v_i^0}{V^1 + V^0} = \dfrac{M(v_i^1, v_i^0)}{M(V^1, V^0)}$

$= \theta w_i^1 + (1-\theta) w_i^0$, where $\theta = \dfrac{V^1}{V^1 + V^0}$

Törnqvist II (1936) : $\bar{w}_i = \frac{1}{2}(w_i^1 + w_i^0) = M(w_i^1, w_i^0)$

Vartia I (1974) : $\bar{w}_i = \dfrac{L(v_i^1, v_i^0)}{L(V^1, V^0)}$, where $L(x,y) = \dfrac{x-y}{\ln x - \ln y}$

= logarithmic mean

Vartia II (1974) : $\bar{w}_i = \dfrac{L(w_i^1, w_i^0)}{\Sigma L(,)}$

Theil (1973) : $\bar{w}_i = \dfrac{T(w_i^1, w_i^0)}{\Sigma T(,)}$, where $T(x,y) = \sqrt[3]{xy\left(\frac{x+y}{2}\right)}$

Vartia III (1974) : $H(P_0^1) = \Sigma\left(\dfrac{v_i^1 + v_i^0}{V^1 + V^0}\right) H\left(\dfrac{p_i^1}{p_i^0}\right)$, where $H\left(\frac{y}{x}\right) = \dfrac{y-x}{\frac{1}{2}(y+x)}$ is the indicator of relative change