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IRREVERSIBLE INVESTMENT UNDER INTEREST RATE VARIABILITY: SOME GENERALIZATIONS

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ABSTRACT: The current extensive literature on irreversible investment decisions usually makes the assumption of constant interest rate. In this paper we study the impact of interest rate and revenue variability on the decision to carry out an irreversible investment project. Given the generality of the considered valuation problem, we first provide a thorough mathematical characterization of the two-dimensional optimal stopping problem and develop some new results. We establish that interest rate variability has a profound decelerating or accelerating impact on investment demand depending on whether the current interest rate is below or above the long run steady state interest rate and that its quantitative size may be very large. Moreover, allowing for interest rate uncertainty is shown to decelerate rational investment demand by raising both the required exercise premium of the irreversible investment opportunity and the value of waiting. Finally, we demonstrate that increased revenue volatility strengthens the negative impact of interest rate uncertainty and vice versa.

KEYWORDS: Irreversible investment, variable interest rates, free boundary problems.

JEL Subject Classification: Q23, G31, C61

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1 Introduction

Most major investments are at least partly irreversible in the sense that firms cannot disinvest. This is because most capital is industry- or firm-specific so that it cannot be used in a different industry or by a different firm. Even though investment would not be firm- or industry-specific, they still could be partly irreversible because of the "lemons" problem meaning that their resale value is often below their purchase cost (cf. Dixit and Pindyck 1994, pp.8-9). Since the seminal work by Arrow 1968 and Nickell 1974, 1978, who analysed irreversible investments under certainty, decisions about irreversible investments in the presence of various types of uncertainties have been studied extensively (see e.g. Baldursson and Karatzas 1997, Baldwin 1982, Bertola and Caballero 1994, Caballero 1991, Demers 1991, Hartman and Hendrickson 2002, Henry 1974, Hu and Oksendal 1998, Kobila 1993, McDonald and Siegel 1986, Oksendal 2001, and Pindyck, 1998, 1991 and Sarkar 2000). In these studies option pricing techniques have been used to show that in the presence of uncertainty the irreversible investment is undertaken when the net present value is "sufficiently high" compared with the opportunity cost. Moreover, even small sunk costs may produce a wide range of inaction. Bernanke 1983 and Cukierman 1980 have developed related models, where firms has an incentive to postpone irreversible investment because doing this they can wait for new information to arrive. The various approaches and applications are excellently reviewed and extended in the seminal book by Dixit and Pindyck 1994. For a more recent review, see Bertola 1998.

In the studies mentioned above, which deal with the impact of irreversibility in a variety of problems and different types of frameworks, the discount rate has assumed to be constant. A motivation for this assumption has been to argue that interest rates are typically more stable and consequently less important than the revenue dynamics. As Dixit and Pindyck 1994 state:

> "Once we understand why and how firms should be cautious when deciding whether to exercise their investment options, we can also understand why

interest rates seem to have so little effect on investment. (p. 13)"

"Second, if an objective of public policy is to stimulate investment, the stability of interest rates may be more important than the level of interest rates. (p. 50)"

Although this argumentation is undoubtedly correct to short-lived investment projects, many real investment opportunities have considerably long planning and exercise periods, which implies that the assumed constancy of the interest rate is questionable. This observation raises several questions: Does interest rate variability matter and, if so, in what direction and how much? What is the role of stochastic interest rate volatility from the point of view of exercising investment opportunities?

Ingersoll and Ross 1992 have studied the role of variability and stochasticity of interest rate on investment decisions. While they also discuss a more general case, in their model they, however, emphasize the role of interest rate uncertainty and consequently specify the interest rate process as a martingale, i.e. as a process with no drift. It is known on the basis of extensive empirical research both that interest rates fluctuate a lot over time and that in the long run interest rates follow a more general mean reverting process (for an up-to-date theoretical and empirical surveys in the filed, see e.g. Bjrk 1998, ch 17, and Cochrane 2001, ch 19). Since variability of interest rates may be deterministic and/or stochastic, we immediately observe that interest rate variability can in general be important from the point of view of exercising real investment opportunities. Motivated by this argumentation from the point of view of long-lived investments, we generalize the important findings by Ingersoll and Ross 1992 in the following respects. First, we allow for stochastic interest rate of a mean reverting type and second, we explore the interaction between stochastic interest rate and stochastic revenue dynamics in terms of the value and the optimal exercise policy of irreversible real investment opportunities.

We proceed as follows. We start our analysis in section 2 by considering the case where both the revenue and interest rate dynamics are variable, but deterministic. After providing a technical characterization of the problem we demonstrate that when the current interest rate is above (below) the long run steady state interest rate, then investment strategies based on the usual assumption of constant discounting will underestimate (overestimate) the value of waiting and the required exercise premium of the irreversible investment policy. We also show a new, though natural, result according to which differences between the required exercise premiums with variable and constant discounting become smaller as the rate of change of interest rate process over time diminishes. In section 3 we extend our model to cover the situation, where the underlying interest rate dynamics is stochastic and demonstrate that interest rate uncertainty strengthens the effect of interest rate variability on the value of waiting and optimal exercise policy. Section 4 further extends the analysis by allowing the revenue dynamics to follow a geometric Brownian motion. We demonstrate that revenue uncertainty strengthens the negative impact of interest rate uncertainty and vice versa. Finally, there is a brief concluding section.

2 Irreversible Investment with Deterministic Interest Rate

In this section we consider the determination of an optimal irreversible investment policy in the presence of deterministic interest rate variability. This provides a good intuitive explanation for the simplest case of a non-constant discount rate. We proceed as follows: First, we provide a set of sufficient conditions under which the optimal exercise date of investment opportunity can be solved generally and in an interesting special case even explicitly. Second, we demonstrate the relationship between the optimal exercise dates with variable and constant discounting when the interest rate can be below or above the long-run steady state interest rate. Finally, we show that the value of investment opportunity is a decreasing and convex function of the current interest rate which will be generalized later on for the stochastic interest rate case as well.

In order to accomplish these tasks, we describe the underlying dynamics for the value of investment X_t and the interest rate r_t as

$$X_t' = \mu X_t, \quad X_0 = x \tag{2.1}$$

and

$$r_t' = \alpha r_t(1-\beta r_t), \quad r_0 = r, \tag{2.2}$$

where μ, γ, α, and β are exogenously determined positive constants. That is, we assume that the revenues accrued from exercising the irreversible investment opportunity increase at an exponential rate and that the interest rate dynamics follow a logistic dynamical system which is consistent with the empirically plausible notion that the interest rate is a mean reverting process. As usually, we denote as

$$\mathcal{A} = \mu x \frac{\partial}{\partial x} + \alpha r(1-\beta r)\frac{\partial}{\partial r}$$

the differential operator associated with the inter-temporally time homogeneous two-dimensional process (X_t, r_t).

Given these assumptions, we now consider the optimal irreversible investment problem

$$V(x,r) = \sup_{t\geq 0}\left[e^{-\int_0^t r_s ds}(X_t - c)\right], \tag{2.3}$$

where c is the sunk cost of investment. As usually in the literature on real options, the determination of the optimal exercise date of the irreversible investment policy can be viewed as the valuation of a perpetual American forward contract on a dividend paying asset. However, *in contrast to previous models relying on constant interest rates, the valuation is now subject to a variable interest rate and, therefore, constitutes a two-dimensional optimal stopping problem.* The continuous differentiability of the exercise payoff implies that (2.3) can also be restated as (cf. Øksendal 1998)

$$V(x,r) = (x-c) + F(x,r), \tag{2.4}$$

where

$$F(x,r) = \sup_{t\geq 0}\int_0^t e^{-\int_0^s r_y dy}[\mu X_s - r_s(X_s - c)]ds \tag{2.5}$$

is known as *the early exercise premium of the considered irreversible investment opportunity*. We now establish the following.

Theorem 2.1. *Assume that $1 > \beta\mu$, so that the percentage growth rate μ of the revenues X_t is below the long run steady state β^{-1} of the interest rate r_t. Then, for all $(x,r) \in C = \{(x,r) \in \mathbb{R}^2_+ : rc > (r-\mu)x\}$ the optimal exercise date of the investment opportunity $t^*(x,r) = \inf\{t \geq 0 : r_t c - (r_t - \mu)X_t \leq 0\}$ is finite and the value $V(x,r)$ constitutes the solution of the boundary value problem*

$$
\begin{aligned}
&(\mathcal{A}V)(x,r) - rV(x,r) = 0 \quad (x,r) \in C \\
&V(x,r) = x - c, \quad \frac{\partial V}{\partial x}(x,r) = 1, \quad \frac{\partial V}{\partial r}(x,r) = 0 \quad (x,r) \in \partial C.
\end{aligned}
$$

Proof. See Appendix A. □

Theorem 2.1 states a set of sufficient conditions under which the optimal investment problem (2.3) can be solved in terms of the initial states (x,r) and the exogenous variables. The non-linearity of the optimality condition implies that it is typically very difficult, if possible at all, to solve explicitly the optimal exercise date of the investment opportunity in the general case. Fortunately, there is an interesting special case under which the investment problem can be solved explicitly. This case is treated in the following.

Corollary 2.2. *Assume that $1 > \beta\mu$ and that $\mu = \alpha$. Then, for all $(x,r) \in C = \{(x,r) \in \mathbb{R}^2_+ : rc > (r-\mu)x\}$ the optimal exercise date of the investment opportunity is*

$$t^*(x,r) = \frac{1}{\mu}\ln\left(1 + \frac{rc - (r-\mu)x}{rx(1-\mu\beta)}\right).$$

In this case, the value reads as

$$V(x,r) = \begin{cases} x - c & (x,r) \in \mathbb{R}^2_+ \backslash C \\ \frac{\mu x}{r}\left(\frac{x - \beta r(x-c)}{x(1-\mu\beta)}\right)^{1-1/(\mu\beta)} & (x,r) \in C. \end{cases} \tag{2.6}$$

Moreover,

$$\frac{\partial t^*}{\partial x}(x,r) = -\frac{rc}{\mu x(rx(1-\mu\beta) + rc - (r-\mu)x)} < 0$$

and

$$\frac{\partial t^*}{\partial r}(x,r) = -\frac{x}{r(rx(1-\mu\beta)+rc-(r-\mu)x)} < 0.$$

Proof. See Appendix B. □

Corollary 2.2 shows that whenever the percentage growth rates at low values (i.e. as $(X_t, r_t) \to (0,0)$) of the revenue and interest rate process coincide, i.e. when $\mu = \alpha$, then both the value and the optimal exercise date of the irreversible investment policy can be solved explicitly in terms of the current states (x, r) and the exogenous variables of the problem. The optimal exercise date is a decreasing function of the initial states x and r. The intuition is obvious. Both higher interest rate and higher revenue increase the opportunity cost of waiting and thereby make the optimal exercise date earlier. Another important implication of our Theorem 2.1 demonstrates how the value and the optimal exercise date of our problem are related to their counterparts under constant discounting. This relationship is summarized in the following.

Corollary 2.3. *Assume that the conditions $1 > \beta\mu$ and $r > \mu$ are satisfied. Then,*

$$\lim_{\alpha\downarrow 0} V(x,r) = x^{r/\mu} \sup_{y\geq x}\left[\frac{y-c}{y^{r/\mu}}\right] = \tilde{V}(x,r), \tag{2.7}$$

and

$$\lim_{\alpha\downarrow 0} t^*(x,r) = \frac{1}{\mu}\ln\left(\frac{rc}{(r-\mu)x}\right) = \tilde{t}(x,r), \tag{2.8}$$

where $\tilde{V}(x,r) = \sup_{t\geq 0}[e^{-rt}(X_t - c)]$ denotes the value and $\tilde{t}(x,r)$ the optimal exercise date under constant discounting, respectively.

Proof. The alleged results are direct consequences of the proof of our Theorem 2.1. □

Remark: It is worth observing that the value of the optimal investment policy in the presence of a constant interest rate can also be expressed as

$$\tilde{V}(x,r) == \begin{cases} x-c & x \geq rc/(r-\mu) \\ \frac{\mu x}{r}\left(\frac{rc}{(r-\mu)x}\right)^{1-r/\mu} & x < rc/(r-\mu) \end{cases}.$$

According to Corollary 2.3 the value and the optimal exercise date of the investment policy in the presence of interest rate variability tend towards their counterparts in the presence of constant discounting as the growth rate of the interest rate process tends to zero. This means naturally that *if the interest rate process evolves towards its long run steady state β^{-1} at a very slow rate, then the conclusions obtained in models neglecting interest rate variability will not be grossly in error when compared with the predictions obtained in models taking into account the variability of interest rates.* In order to illustrate the potential quantitative role of these qualitative differences we next provide some numerical computations. In Table 1 we have used the assumption that $c = 1$, $\mu = 1\%$, $\beta^{-1} = 3\%$, $r = 5\%$ and $x = 0.1$ (implying that $\tilde{t}(0.1, 0.05) = 91.6291$). Hence, in this case the long-run steady state of interest is below the current interest rate. As Table 1 and Figure 1 illustrate, interest rate variability affects both the exercise date and the value of waiting.

α	$t^*(0.1, 0.05)$	$X(t^*(0.1, 0.05)) - c$
5%	109.779	0.498761
1%	102.962	0.4
0.5%	98.3206	0.336506
10^{-6}	91.6306	0.250019

Table 1: The Optimal Exercise Date and Required Exercise Premium.

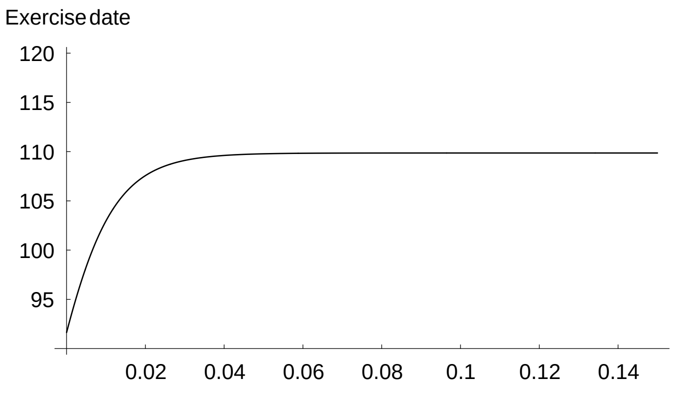

Figure 1: The Optimal Exercise Date $\tilde{t}(0.1, 0.05)$ as a function of α

In Table 2 we illustrate our results under the assumption that the long-run steady state interest rate is above the current interest rate. More precisely, we assume that $c = 1$, $\mu = 1\%$, $\beta^{-1} = 3\%$, $r = 1.5\%$ and $x = 0.1$ (implying that $\tilde{t}(0.1, 0.015) = 179.176$). In this case interest rate variability has the reverse effect on the exercise date and the value of waiting than in the case where the steady state interest rate is below the current rate of interest.

α	$t^*(0.1, 0.015)$	$X(t^*(0.1, 0.015)) - c$
5%	110.065	0.503061
1%	125.276	0.75
0.5%	138.629	1
10^{-6}	179.158	1.99946

Table 2: The Optimal Exercise Date and Required Exercise Premium.

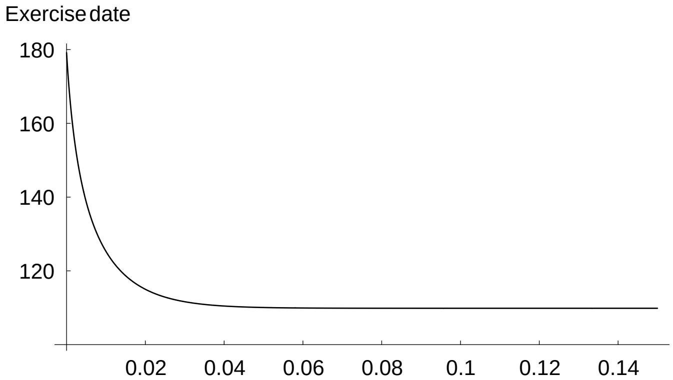

Figure 2: The Optimal Exercise Date $\tilde{t}(0.1, 0.015)$ as a function of α

After having characterized a set of conditions under which the optimal investment problem with variable discounting can be solved in terms of the initial states of the system and exogenous variables and having provided explicit solutions in an interesting special case, we now ask the following important but, to our knowledge, thus far unexplored question: What is the relationship between the optimal exercise policy and the value of the investment opportunity with variable and constant discounting. Given the definitions of the optimal policy and its value under the deterministic evolution of the interest rate, we are now in the position to establish the following new set of results summarized in

Theorem 2.4. *Assume that* $1 > \beta\mu$ *and that* $r > \mu$. *Then,*

$$t^*(x,r) \gtreqqless \tilde{t}(x,r), \quad V(x,r) \gtreqqless \tilde{V}(x,r) \quad \textit{and} \quad F(x,r) \gtreqqless \tilde{F}(x,r) \quad \textit{when} \quad r \gtreqqless \beta^{-1}.$$

Proof. See Appendix C. □

Theorem 2.4 generalizes the finding by Ingersoll and Ross 1992 (pp.4–5) by characterizing the differences of the optimal exercise policy and the value of the investment opportunity with constant and variable discounting. First, the required exercise premium and the value of the investment opportunity is higher in the presence of variable discounting than under constant discounting when the current interest rate is above the long-run steady state interest rate. Second, the reverse happens when the current interest rate is below the long-run steady state interest rate. More specifically, these findings imply the following important finding: *When the current interest rate is above (below) the long run steady state value, then the investment strategies based on the usual approach neglecting the interest rate variability will underestimate (overestimate) both the value of waiting and the required exercise premium of the irreversible investment policy.*

Theorem 2.4 characterizes qualitatively the differences of the optimal exercise policy and the value of investment opportunities with constant and variable discounting. In Figure 3, we illustrate these findings quantitatively in an example where the steady state interest rate $\hat{r}$ is 3% and the current interest rate is either above the steady state interest rate (the l.h.s. of Figure 3) or below the steady state interest rate (the r.h.s. of Figure 3). The other parameters are $c = 1, \mu = 1\%$, and $\beta^{-1} = 3\%$. The solid lines describe the exercise dates in the presence of variable interest rate while the dotted lines the optimal exercise dates with constant discounting. One can see from Figure 3 that when the current interest rate is above the steady state interest rate, the difference between the exercise dates becomes larger the higher is the current interest rate. Naturally, the reverse happens when the current interest rate is below the steady state interest rate.

Hence, the differences between the exercise dates can be very large if the variability of interest rate is big enough.

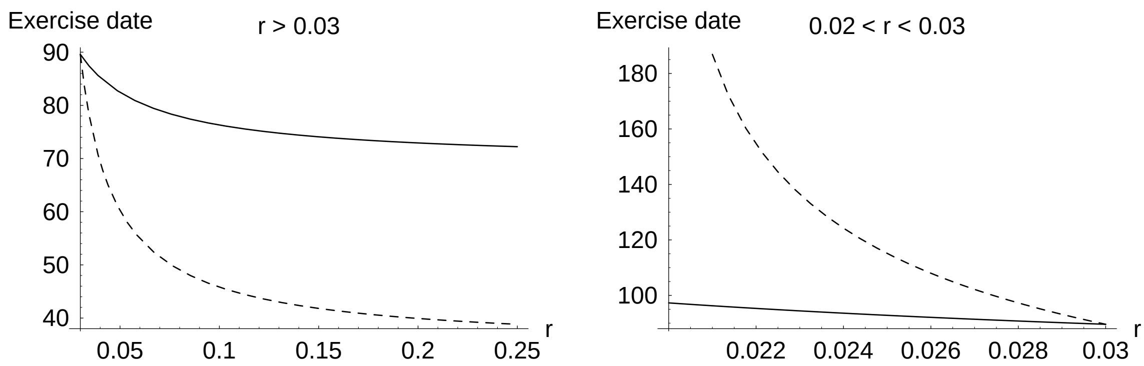

Figure 3: The Optimal Exercise Date $t^*(x, r)$

It is worth observing that if $\alpha = \mu$, then the *required exercise premiums* read in the presence of a variable interest rate as

$$P(x, r) = X_{t^*(x,r)} - c = \frac{\mu c}{\beta^{-1} - \mu}\left[1 + \frac{x(1 - \beta r)}{\beta r c}\right] \tag{2.9}$$

and in the presence of constant discounting as

$$\tilde{P}(x, r) = X_{\tilde{t}(x,r)} - c = \frac{\mu c}{r - \mu}. \tag{2.10}$$

Moreover, as intuitively is clear, $P(x, \beta^{-1}) = \tilde{P}(x, \beta^{-1})$ so that the required exercise premiums coincide at the long run asymptotically stable steady state of the interest rate. As we can observe from (2.9)

$$\frac{\partial P}{\partial x}(x, r) = \frac{\mu c}{\beta^{-1} - \mu}\left[\frac{1 - \beta r}{\beta r c}\right] \gtreqqless 0, \quad r \lesseqqgtr \beta^{-1},$$

and

$$\frac{\partial P}{\partial r}(x, r) = -\frac{\mu c}{\beta^{-1} - \mu}\left[\frac{x}{r^2 \beta c}\right] < 0.$$

Hence, the *required exercise premium is a decreasing function of the current interest rate r at all states, while the sign of the sensitivity of the required exercise premium is positive (negative) provided that the current interest rate r is below (above) the long run* *steady state* β^{-1}. Before proceeding further in our analysis, we prove the following result characterizing the monotonicity and curvature properties of the value of the investment opportunity.

Lemma 2.5. *Assume that the conditions of Theorem 2.1 are satisfied. Then, the value of the investment opportunity is an increasing and convex function of the current revenues x and a decreasing and convex function of the current interest rate r.*

Proof. See Appendix D. □

Later on we generalize these properties to cover the case of stochastic interest rate and stochastic revenue.

3 Irreversible Investment with Interest Rate Uncertainty

In the analyzes we have carried out thus far, the underlying dynamics for the revenue X_t and the interest rate r_t has been postulated to be deterministic. The reason for this was that we first wanted to show the impact of variable discounting on the investment decisions in the simpler case to provide an easy intuition. In this section we generalize our earlier analysis by exploring the optimal investment decision in the presence of interest rate uncertainty. We proceed as follows. First, we characterize a set of sufficient conditions for the optimality of investment strategy and second, we show how under certain plausible conditions the interest rate uncertainty has the impact of postponing the optimal exercise of investment opportunity.

We assume that the interest rate process $\{r_t; t \geq 0\}$ is defined on a complete filtered probability space $(\Omega, P, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F})$ satisfying the usual conditions and that r_t is described on $\mathbb{R}_+$ by the (Itô-) stochastic differential equation of a mean reverting type

$$dr_t = \alpha r_t(1 - \beta r_t)dt + \sigma r_t dW_t, \quad r_0 = r, \tag{3.1}$$

where $\sigma > 0$ is an exogenously determined parameter measuring the volatility of the underlying interest rate dynamics. This kind of specification lies in conformity with empirics (see, e.g. Cochrane 2002, ch 19) and can also be theoretically supported (cf. Merton 1975). It is now clear that given our assumptions on the underlying dynamics the differential operator associated with the two-dimensional process (X_t, r_t) now reads as

$$\hat{\mathcal{A}} = \frac{1}{2}\sigma^2 r^2 \frac{\partial^2}{\partial r^2} + \mu x \frac{\partial}{\partial x} + \alpha r(1-\beta r)\frac{\partial}{\partial r}.$$

Applying Itô's lemma to the mapping $r \mapsto \ln r$ yields that

$$e^{-\int_0^t r_s ds} = \left(\frac{r_t}{r}\right)^{\frac{1}{\alpha\beta}} e^{-\frac{1}{\beta}t + \frac{\sigma^2}{2\alpha\beta}\left(1+\frac{1}{\alpha\beta}\right)t} M_t, \tag{3.2}$$

where $M_t = e^{-\frac{\sigma}{\alpha\beta}W_t - \frac{\sigma^2}{2\alpha^2\beta^2}t}$ is a positive exponential $\mathcal{F}_t$-martingale. According to equation (3.2) the discount factor can be expressed in a path independent form which only depends on both the initial r and the current interest rate r_t. It is worth emphasizing that if $\alpha > \sigma^2/2$, then the interest rate process r_t converges towards a long run stationary distribution with density (a χ^2-*distribution*, cf. Alvarez and Shepp 1998)

$$p(r) = \left(\frac{2\alpha\beta}{\sigma^2}\right)^{\frac{\rho}{2}} \frac{r^{\frac{(\rho-2)}{2}} e^{-\frac{2\alpha\beta r}{\sigma^2}}}{\Gamma(\rho/2)},$$

where $\rho/2 = \frac{2\alpha}{\sigma^2} - 1 > 0$. Given this distribution, the expected long-run interest rate reads as

$$\lim_{t\to\infty} \mathbf{E}[r_t] = \left(1 - \frac{\sigma^2}{2\alpha}\right)\frac{1}{\beta} < \frac{1}{\beta}$$

and satisfies the intuitively clear condition

$$\frac{\partial}{\partial\sigma}\lim_{t\to\infty} \mathbf{E}[r_t] = -\frac{\sigma}{\alpha\beta} < 0$$

meaning that higher interest rate volatility decreases the expected value of the expected steady state interest rate.

Given these plausible technical assumptions, we now consider the valuation of the irreversible investment opportunity in the presence of interest rate uncertainty. More precisely, we consider the optimal stopping problem

$$\hat{V}_\sigma(x,r) = \sup_\tau \mathbf{E}_{(x,r)}\left[e^{-\int_0^\tau r_s ds}(X_\tau - c)\right], \tag{3.3}$$

where τ is an arbitrary $\mathcal{F}_t$-stopping time. It is worth mentioning that we apply the notation $\hat{V}_\sigma(x,r)$ in order to emphasize the dependence of the value of the optimal policy on the volatility of the underlying interest rate process. In line with our results of the previous section, Dynkin's theorem (cf. Øksendal 1998, pp. 118-120) implies that the optimal stopping problem (3.3) can also be rewritten as in (2.4) with the exception that the early exercise premium now reads as

$$\hat{F}_\sigma(x,r) = \sup_\tau \mathbf{E}_{(x,r)} \int_0^\tau e^{-\int_0^s r_y dy}(\mu X_s - r_s(X_s - c))ds. \tag{3.4}$$

This type of path dependent optimal stopping problems are typically studied by relying on a set of variational inequalities characterizing the value of the associated free boundary problem (cf. Øksendal and Reikvam 1998). Unfortunately, multi-dimensional optimal stopping problems of the type (3.3) are extremely difficult, if possible at all, to be solved explicitly in terms of the current states and the exogenous parameters of the problem.

However, given (3.2) and defining the equivalent martingale measure $\mathbb{Q}$ through the likelihood ratio $d\mathbb{Q}/d\mathbb{P} = M_t$ we now find importantly that the two dimensional path-dependent optimal stopping problem (3.3) can be re-expressed as

$$\hat{V}_\sigma(x,r) = r^{-\frac{1}{\alpha\beta}} \sup_\tau \mathbf{E}_{(x,r)} \left[e^{-\theta\tau} \tilde{r}_\tau^{\frac{1}{\alpha\beta}} (X_\tau - c) \right], \tag{3.5}$$

where $\theta = \frac{1}{\beta} - \frac{\sigma^2}{2\alpha\beta}\left(1 + \frac{1}{\alpha\beta}\right)$ and the diffusion $\tilde{r}_t$ evolves according to the dynamics described by the stochastic differential equation

$$d\tilde{r}_t = \alpha\tilde{r}_t \left(1 - \frac{\sigma^2}{\alpha^2\beta} - \beta\tilde{r}_t\right) dt + \sigma\tilde{r}_t dW_t, \quad \tilde{r}_0 = r. \tag{3.6}$$

It is worth observing that the optimal stopping problem (3.5) is path-independent and, thus, typically easier to handle than the original problem (3.3). An important requirement (the so-called absence of speculative bubbles condition) guaranteeing the finiteness of the considered valuation is that

$$\frac{1}{\beta} > \mu + \frac{\sigma^2}{2\alpha\beta}\left(1 + \frac{1}{\alpha\beta}\right),$$

which is naturally a stronger requirement than the condition $1 > \beta\mu$ required in the deterministic case.

We can now establish a qualitative connection between the deterministic and stochastic stopping problems (2.3) and (3.3). This is summarized in the following theorem which could be called *the fundamental qualitative characterization of the value of an irreversible investment opportunity in the presence of interest rate uncertainty.*

Theorem 3.1. *Assume that $\theta > \mu$. Then interest rate uncertainty increases both the required exercise premium and the value of the irreversible investment opportunity and, consequently, postpones the optimal exercise of investment opportunities.*

Proof. See Appendix E. □

This new result shows that under a set of plausible assumptions both the value and the optimal exercise boundary of the investment opportunity is higher in the presence of interest rate uncertainty than in its absence. It would be of interest to characterize more precisely the difference between the optimal policy in the absence of uncertainty with the optimal policy in the presence of uncertainty. Unfortunately, stopping problems of the type (3.3) are seldom solvable and, consequently, the difference between the optimal policies can typically be illustrated only numerically.

Before establishing the sign of the relationship between interest rate volatility and investment, we first present an important result characterizing the form of the value function $\hat{V}(x, r)$ as a function of the current revenues x and the current interest rate r. This is accomplished in the following.

Lemma 3.2. *The value function $\hat{V}_\sigma(x, r)$ is an increasing and convex function of the current revenues x and a decreasing and convex function of the current interest rate r.*

Proof. See Appendix F. □

Lemma 3.2 is very important since it implies that the sign of the relationship between interest rate volatility and investment in unambiguously negative and it suggests a generalization of the findings by Ingersoll and Ross 1992 where they characterize the impact of riskiness of the interest rate path on the value of waiting (see Theorem on p. 26). More precisely, we have

Theorem 3.3. *Increased interest rate volatility increases both the value and the early exercise premium of the irreversible investment opportunity. Moreover, it also expands the continuation region and, therefore, postpones the optimal exercise of irreversible investment opportunities.*

Proof. See Appendix G. □

According to Theorem 3.3, more volatile interest dynamics leads to postponement of investment because of the convexity of the value function. An economic interpretation goes as follows. Increased interest rate volatility that the opportunity cost of not investing becomes more uncertain, which will move the exercise date further into the future. While increased volatility increases the expected present value of future revenues, it simultaneously increases the value of holding the opportunity alive. Since the latter effect dominates, the net effect of increased volatility is to postpone the optimal exercise of investment opportunities (cf. Dixit and Pindyck 1994).

4 Irreversible Investment with Interest Rate and Revenue Uncertainty

After having characterized the relationship between the value and optimal exercise of investment opportunities when the underlying interest rate dynamics was assumed to be stochastic and the revenue dynamics was deterministic, we extend the analysis of the previous section. We now assume that the interest rate dynamics follow the diffusion described by the stochastic differential equation (3.1) and that the revenue dynamics, instead of being deterministic, is described on $\mathbb{R}_+$ by the stochastic differential equation

$$dX_t = \mu X_t dt + \gamma X_t d\bar{W}_t \quad X_0 = x, \tag{4.1}$$

where $\bar{W}_t$ is a Brownian motion independent of W_t and $\mu > 0$, $\gamma > 0$ are exogenously given constants. It is clear that given the stochasticity of the revenue dynamics, the differential operator associated with the process (X_t, r_t) now reads as

$$\bar{\mathcal{A}}_\gamma = \frac{1}{2}\sigma^2 r^2 \frac{\partial^2}{\partial r^2} + \frac{1}{2}\gamma^2 x^2 \frac{\partial^2}{\partial x^2} + \mu x \frac{\partial}{\partial x} + \alpha r(1-\beta r)\frac{\partial}{\partial r}.$$

Given the dynamics of the process (X_t, r_t) we now plan to consider the following optimal stopping problem

$$\bar{V}_{\sigma,\gamma}(x,r) = \sup_\tau \mathbf{E}_{(x,r)}\left[e^{-\int_0^\tau r_s ds}(X_\tau - c)\right], \tag{4.2}$$

where τ is an arbitrary stopping time. Again, we find that defining the equivalent martingale measure $\mathbb{Q}$ through the likelihood ratio $d\mathbb{Q}/d\mathbb{P} = M_t$ implies that the path dependent optimal stopping problem (4.2) can be re-expressed as

$$\bar{V}_{\sigma,\gamma}(x,r) = r^{-\frac{1}{\alpha\beta}} \sup_\tau \mathbf{E}_{(x,r)}\left[e^{-\theta\tau}\tilde{r}_\tau^{\frac{1}{\alpha\beta}}(X_\tau - c)\right], \tag{4.3}$$

where θ and $\tilde{r}_t$ are defined as in the previous section. Observing finally that $X_t = xe^{\mu t}\bar{M}_t$, where $\bar{M}_t = e^{\gamma\bar{W}_t - \frac{1}{2}\gamma^2 t}$ is a positive exponential martingale again implies that the value (4.2) is finite provided that the absence of speculative bubbles condition $\theta > \mu$ is satisfied (otherwise the first term of the value would explode as $t \to \infty$). In line with our previous findings, we can now establish the following.

Lemma 4.1. *The value of the investment opportunity is an increasing and convex function of the current revenues and an increasing and convex function of the current interest rate.*

Proof. It is now clear that the solution of the stochastic differential equation (4.1) is $X_t = xe^{\mu t}M_t$, where $M_t = e^{\gamma\bar{W}(t) - \gamma^2 t/2}$ is a positive exponential martingale. Consequently, all the elements in the sequence of value functions $V_n(x,r)$ presented in the proof of Lemma 3.2 are increasing and convex as functions of the current revenues x (cf. El Karoui, Jeanblanc-Picqué, and Shreve 1998). This implies that the value function is increasing and convex as a function of the current revenues x. The rest of the proof is analogous with the proof of Lemma 3.2. □

The key implication of Lemma 4.1 is now presented in

Theorem 4.2. *Assume that the conditions of Lemma 4.1 are satisfied, and that $\theta > \mu$. Then, increased interest rate or revenue volatility increases both the value and the early exercise premium of the optimal policy. Moreover, increased interest rate or revenue volatility expands the continuation region and, thus, postpones the optimal exercise of investment opportunities.*

Proof. The proof is analogous with the proof of Theorem 3.1. □

Theorem 4.2 shows that revenue uncertainty strengthens the negative effect of interest rate uncertainty and vice versa. Put somewhat differently, Theorem 4.2 shows that the combined impact of interest rate and revenue uncertainty dominates the impact of individual interest rate and individual revenue uncertainty. Consequently, our results verify the intuitively clear result that uncertainty, independently of its source, slows down rational investment demand by increasing the required exercise premium of a rational investor. It is also worth emphasizing that given the convexity of the value function, combined interest rate and revenue volatility will increase the value and the required exercise threshold in comparison with the case in the absence of revenue uncertainty.

5 Conclusions

In this paper we have considered the determination of an optimal irreversible investment policy with variable discounting and demonstrated several new results. We started our analysis by considering the case of deterministic interest rate variability. First, we provided a set of sufficient conditions under which this two-dimensional optimal stopping problem can be solved generally and in an interesting special case explicitly. Second, we demonstrated the relationship between the optimal exercise dates with variable and constant discounting when the interest rate can be below or above the long-run steady state interest rate. Third, we showed that the value of the investment opportunity is an increasing and convex function of the current revenues and a decreasing and convex function of the current interest rate.

We have also generalized our deterministic analysis in two important respects. First, we have explored the optimal investment decision in the presence of interest rate uncertainty, i.e. when the interest rate process is of a mean reverting type, which lien in conformity with empirics, but fluctuates stochastically, and second, we have allowed for revenue dynamics to follow geometric Brownian motion. In this setting we characterized a set of sufficient conditions which can be applied for the verification of the optimality of an investment strategy. Moreover, we have showed how under certain plausible conditions the interest rate uncertainty postpones the rational exercise of investment opportunity. Finally, and importantly, we demonstrated that revenue uncertainty strengthens the negative impact of interest rate uncertainty and vice versa.

An interesting area for further research would be to examine the effects of taxation in the presence of potentially stochastically dependent revenue and interest rate uncertainty. Such an analysis has not been done, and, is out of the scope of the present study and is, therefore, left for future research.

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A Proof of Theorem 2.1

Proof. It is a simple exercise in ordinary analysis to demonstrate that

$$X_t = xe^{\mu t}, \quad r_t = \frac{re^{\alpha t}}{1+\beta r(e^{\alpha t}-1)},$$

$$e^{-\int_0^t r_s ds} = \left(1+\beta r(e^{\alpha t}-1)\right)^{-1/(\alpha\beta)},$$

and that

$$\frac{d}{dt}\left[e^{-\int_0^t r_s ds}(X_t - c)\right] = e^{-\int_0^t r_s ds}(\mu X_t - r_t(X_t - c)). \tag{A.1}$$

Given the solutions of the ordinary differential equations (2.1) and (2.2), we observe that (A.1) can be rewritten as

$$(1+\beta r(e^{\alpha t}-1))) e^{\int_0^t r_s ds}\frac{d}{dt}\left[e^{-\int_0^t r_s ds}(X_t - c)\right] = \mu x(1-\beta r) + rce^{(\alpha-\mu)t} - rx(1-\beta\mu)e^{\alpha t}.$$

Consider now the mapping $f : \mathbb{R}_+ \mapsto \mathbb{R}$ defined as

$$f(t) = \mu x(1-\beta r) + rce^{(\alpha-\mu)t} - rx(1-\beta\mu)e^{\alpha t}.$$

It is now clear that $f(0) = rc - (r-\mu)x$ and that $\lim_{t\to\infty} f(t) = -\infty$. Moreover, since

$$f'(t) = (\alpha-\mu)rce^{(\alpha-\mu)t} - \alpha rx(1-\beta\mu)e^{\alpha t},$$

we find that $f'(t) < 0$ for all $t \geq 0$ whenever $\alpha \leq \mu$ and, therefore, that for any initial state on C, the optimal stopping date $t^*(x,r)$ satisfying the optimality condition $f(t^*(x,r)) = 0$ exists and is finite (because of the monotonicity and the boundary behavior of $f(t)$). Assume now that $\alpha > \mu$. Then, $f'(0) = (\alpha-\mu)rc - \alpha rx(1-\beta\mu)$ and $\lim_{t\to\infty} f'(t) = -\infty$. Moreover, since

$$f''(t) = (\alpha-\mu)^2 rce^{(\alpha-\mu)t} - \alpha^2 rx(1-\beta\mu)e^{\alpha t},$$

we find that $0 = \operatorname{argmax}\{f(t)\}$ provided that $(\alpha-\mu)rc \leq \alpha rx(1-\beta\mu)$ and that

$$\tilde{t} = \frac{1}{\mu}\ln\left(\frac{(\alpha-\mu)c}{\alpha x(1-\beta\mu)}\right)$$

if $(\alpha-\mu)c > \alpha x(1-\beta\mu)$. However, since

$$f''(\tilde{t}) = -\alpha r x(1-\mu\beta)\mu e^{\alpha\tilde{t}} < 0$$

we find that $f'(t) < 0$ for all $(x,r) \in \mathbb{R}_+^2$ in that case as well and, therefore, that for any initial state on C, the optimal stopping date $t^*(x,r)$ satisfying the optimality condition $f(t^*(x,r)) = 0$ exists and is finite.

Having established the existence and finiteness of the optimal exercise date $t^*(x,r)$ we now have to prove that the value satisfies the boundary value problem. Standard differentiation yields (after simplifications)

$$\frac{\partial V}{\partial x}(x,r) = \Big(1+\beta r(e^{\alpha t^*(x,r)}-1)\Big)^{-1/(\alpha\beta)}$$

and

$$\frac{\partial V}{\partial r}(x,r) = -\Big(1+\beta r(e^{\alpha t^*(x,r)}-1)\Big)^{-1/(\alpha\beta)}\frac{1}{\alpha}(X_{t^*(x,r)}-c)(e^{\alpha t^*(x,r)}-1).$$

Applying these equations then proves that $(\mathcal{A}V)(x,r) - rV(x,r) = 0$ for all C. Moreover, since $t^*(x,r) = 0$ whenever $(x,r) \in \partial C$, we find that $\frac{\partial V}{\partial x}(x,r) = 1$ and $\frac{\partial V}{\partial r}(x,r) = 0$ for all $(x,r) \in \partial C$. Our results on the early exercise premium $F(x,r)$ are direct implications of the definition (2.4). □

B Proof of Corollary 2.2

Proof. As was established in the proof of Theorem 2.1, the optimal exercise date $t^*(x,r)$ is the root of $\mu X_{t^*(x,r)} = r_{t^*(x,r)}(X_{t^*(x,r)} - c)$, that is, the root of the equation

$$\mu x e^{\mu t^*(x,r)}(1+\beta r(e^{\mu t^*(x,r)}-1)) = r e^{\mu t^*(x,r)}(x e^{\mu t^*(x,r)} - c).$$

Multiplying this equation with $e^{-\mu t^*(x,r)}$ and reordering the terms yields

$$rx(\mu\beta - 1)e^{\mu t^*(x,r)} = \mu x(\beta r - 1) - rc$$

from which the alleged result follows by taking logarithms from both sides of the equation. Inserting the optimal exercise date $t^*(x,r)$ to the expression

$$V(x,r) = e^{-\int_0^{t^*(x,r)} r_s ds}(X_{t^*(x,r)} - c)$$

then yields the alleged value. Our conclusions on the early exercise premium $F(x,r)$ then follow directly from (2.4). Finally, the comparative static properties of the optimal exercise date $t^*(x,r)$ can then be established by ordinary differentiation. $\square$

C Proof of Theorem 2.4

Proof. It is clear that $\tilde{t}(x,r)$ satisfies the condition $\mu X_{\tilde{t}(x,r)} = r(X_{\tilde{t}(x,r)} - c)$. Define now the mapping $\hat{f}(t) = \mu X_t - r(X_t - c)$. We then find that

$$\hat{f}(\tilde{t}(x,r)) = \mu X_{\tilde{t}(x,r)} - r_{\tilde{t}(x,r)}(X_{\tilde{t}(x,r)} - c) = (r - r_{\tilde{t}})(X_{\tilde{t}(x,r)} - c) \gtreqqless 0, \quad \text{if } r \gtreqqless \beta^{-1},$$

since $r_t \gtreqqless r$ for all $t \geq 0$ when $r \lesseqqgtr \beta^{-1}$. However, since $\hat{f}(t^*(x,r)) = 0$ we find that $t^*(x,r) \gtreqqless \tilde{t}(x,r)$ when $r \gtreqqless \beta^{-1}$.

Assume that $r < \beta^{-1}$ and, therefore, that $r_t > r$ for all $t \geq 0$. Since $\mu x \frac{\partial \tilde{V}}{\partial x}(x,r) \leq r\tilde{V}(x,r)$ and $\tilde{V}(x,r) \geq g(x)$ for all $x \in \mathbb{R}_+$ we find by ordinary differentiation that

$$\begin{aligned} \frac{d}{dt}\left[e^{-\int_0^t r_s ds}\tilde{V}(X_t,r)\right] &= e^{-\int_0^t r_s ds}\left[\mu X_t \frac{\partial \tilde{V}}{\partial x}(X_t,r) - r_t\tilde{V}(X_t,r)\right] \\ &\leq e^{-\int_0^t r_s ds}\left[r - r_t\right]\tilde{V}(X_t,r) \leq 0 \end{aligned}$$

for all $t \geq 0$. Therefore,

$$\tilde{V}(x,r) \geq e^{-\int_0^t r_s ds}\tilde{V}(X_t,r) \geq e^{-\int_0^t r_s ds} g(X_t)$$

implying that $\tilde{V}(x,r) \geq V(x,r)$ when $r < \beta^{-1}$. The proof in the case where $r > \beta^{-1}$ is completely analogous. The conclusions on the early exercise premiums $F(x,r)$ and $\tilde{F}(x,r)$ follow directly from their definitions. $\square$

D Proof of Lemma 2.5

Proof. Consider first the discount factor $e^{-\int_0^t r_s ds}$. Since

$$e^{-\int_0^t r_s ds} = \left(1 + \beta r(e^{\alpha t} - 1)\right)^{-1/(\alpha\beta)},$$

we find by ordinary differentiation that

$$\frac{d}{dr}\left[e^{-\int_0^t r_s ds}\right] = -\frac{1}{\alpha}\left(1+\beta r(e^{\alpha t}-1)\right)^{-(1/(\alpha\beta)+1)}(e^{\alpha t}-1) < 0$$

and that

$$\frac{d^2}{dr^2}\left[e^{-\int_0^t r_s ds}\right] = \frac{1}{\alpha}\left(\frac{1}{\alpha\beta}+1\right)\left(1+\beta r(e^{\alpha t}-1)\right)^{-(1/(\alpha\beta)+2)}\beta(e^{\alpha t}-1)^2 > 0$$

implying that the discount factor is a decreasing and convex function of the current interest rate. Since the maximum of a decreasing and convex mapping is decreasing and convex, we find that the value is a decreasing and convex function of the current interest rate r. Similarly, since the exercise payoff $X_t - c$ is increasing and linear as a function of the current state x, we find that the maximum, i.e. the value of the opportunity is an increasing and convex function of the initial revenues x (by classical duality arguments of nonlinear programming). □

E Proof of Theorem 3.1

Proof. As was established in Lemma 2.5, the value of the investment opportunity is convex in the deterministic case. Consequently, we find that for all $(x, r) \in C$ we have that

$$(\hat{\mathcal{A}}V)(x,r) - rV(x,r) = \frac{1}{2}\sigma^2 r^2 \frac{\partial^2 V}{\partial r^2}(x,r) \geq 0,$$

since $(\mathcal{A}V)(x,r) - rV(x,r) = 0$ for all $(x,r) \in C$. Let τ_n be a sequence of stopping times converging towards the stopping time $\tau^* = \inf\{t \geq 0 : \mu X_t \leq r_t(X_t - c)\}$. Dynkin's theorem then yields that

$$\mathbf{E}_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds} V(X_{\tau_n}, r_{\tau_n})\right] \geq V(x,r).$$

Letting $n \to \infty$ and invoking the continuity of the value $V(x,r)$ across the boundary ∂C then yields that

$$V(x,r) \leq \mathbf{E}_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds}(X_{\tau_n} - c)\right] \leq \hat{V}_\sigma(x,r)$$

for all $(x, r) \in C$. However, since $V(x, r) = x - c$ on $\mathbb{R}^2_+ \backslash C$ and $\hat{V}_\sigma(x, r) \geq x - c$ for all $x \in \mathbb{R}^2_+$, we find that $\hat{V}_\sigma(x, r) \geq V(x, r)$ for all $x \in \mathbb{R}^2_+$.

Assume that $(x, r) \in C$. Since $\hat{V}_\sigma(x, r) \geq V(x, r) > (x - c)$, we find that $(x, r) \in \{(x, r) \in \mathbb{R}^2_+ : \hat{V}_\sigma(x, r) > x - c\}$ as well and, therefore, that $C \subset \{(x, r) \in \mathbb{R}^2_+ : \hat{V}_\sigma(x, r) > x - c\}$, thus completing the proof. □

F Proof of Lemma 3.2

Proof. To establish the monotonicity and convexity of the value function $\hat{V}_\sigma(x, r)$ as a function of the current revenues x, we first define the increasing sequence $\{V_n(x, r)\}_{n \in \mathbb{N}}$ iteratively as

$$V_0(x, r) = (x - c), \quad V_{n+1}(x, r) = \sup_{t \geq 0} \mathbf{E}_{(x,r)} \left[e^{-\int_0^t r_s ds} V_n(X_t, r_t) \right].$$

It is now clear that since $V_0(x, r)$ is increasing and linear as a function of x and $X_t = xe^{\mu t}$, the value $V_1(x, r)$ is increasing and convex as a function of x by standard duality arguments from nonlinear programming theory. Consequently, all elements in the sequence $\{V_n(x, r)\}_{n \in \mathbb{N}}$ are increasing and convex as functions of x. Since $V_n(x, r) \uparrow \hat{V}_\sigma(x, r)$ as $n \to \infty$ (cf. Øksendal 1998, p. 200) we find that for all $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}_+$ we have that

$$\lambda \hat{V}_\sigma(x, r) + (1 - \lambda)\hat{V}_\sigma(y, r) \geq \lambda V_n(x, r) + (1 - \lambda)V_n(y, r) \geq V_n(\lambda x + (1 - \lambda)y, r).$$

Letting $n \to \infty$ and invoking monotonic convergence then implies that $\lambda \hat{V}_\sigma(x, r) + (1 - \lambda)\hat{V}_\sigma(y, r) \geq \hat{V}_\sigma(\lambda x + (1 - \lambda)y, r)$ proving the convexity of $\hat{V}_\sigma(x, r)$. Similarly, if $x \geq y$ then

$$\hat{V}_\sigma(x, r) \geq V_n(x, r) \geq V_n(y, r) \uparrow \hat{V}_\sigma(y, r), \quad \text{as } n \to \infty$$

proving the alleged monotonicity of $\hat{V}_\sigma(x, r)$ as a function of x. Finally, as was established in Alvarez and Koskela 2001, our assumptions imply that the discount factor $e^{-\int_0^t r_s ds}$ is an almost surely decreasing and strictly convex function of the current interest rate r and, consequently, that the value function is decreasing and strictly convex as a function of the current interest rate r. □

G Proof of Theorem 3.3

Proof. We know from Lemma 3.2 that given our assumptions, the value $\hat{V}_\sigma(x,r)$ is convex in r. Consequently, we find that for all $(x,r) \in \mathbb{R}^2_+$ we have that

$$(\hat{\mathcal{A}}\hat{V}_{\hat{\sigma}})(x,r) - r\hat{V}_{\hat{\sigma}}(x,r) \leq \frac{1}{2}(\sigma^2 - \hat{\sigma}^2)r^2 \frac{\partial^2 \hat{V}_{\hat{\sigma}}}{\partial r^2}(x,r) \leq 0$$

since

$$\frac{1}{2}\hat{\sigma}^2 r^2 \frac{\partial^2 \hat{V}_{\hat{\sigma}}}{\partial r^2}(x,r) + \mu x \frac{\partial \hat{V}_{\hat{\sigma}}}{\partial x}(x,r) + \alpha r(1-\beta r)\frac{\partial \hat{V}_{\hat{\sigma}}}{\partial r}(x,r) - r\hat{V}_{\hat{\sigma}}(x,r) \leq 0$$

for all $(x,r) \in \mathbb{R}^2_+$ by the r-excessivity of $\hat{V}_{\hat{\sigma}}(x,r)$. Consequently, applying Dynkin's theorem yields that

$$\mathbf{E}_{(x,r)}\left[e^{-\int_0^{\tau_n} r_s ds}\hat{V}_{\hat{\sigma}}(X_{\tau_n}, r_{\tau_n})\right] \leq \hat{V}_{\hat{\sigma}}(x,r)$$

where $\tau_n = \tau \wedge n \wedge \inf\{t \geq 0 : \sqrt{X_t^2 + r_t^2} > n\}$ is an almost surely finite stopping time and r_t denote the interest rate process subject to the less volatile dynamics. Reordering terms, invoking the condition $\hat{V}_{\hat{\sigma}}(x,r) \geq (x-c)$, letting $n \to \infty$, and applying Fatou's theorem yields that

$$\hat{V}_{\hat{\sigma}}(x,r) \geq \mathbf{E}_{(x,r)}\left[e^{-\int_0^{\tau} r_s ds}(X_\tau - c)\right]$$

proving that $\hat{V}_{\hat{\sigma}}(x,r) \geq \hat{V}_\sigma(x,r)$ for all $(x,r) \in \mathbb{R}^2_+$. The inequality $\hat{F}_{\hat{\sigma}}(x,r) \geq \hat{F}_\sigma(x,r)$ then follows from the definition of the early exercise premiums. Finally, if $(x,r) \in \{(x,r) \in \mathbb{R}^2_+ : \hat{V}_\sigma(x,r) > (x-c)\}$, then $(x,r) \in \{(x,r) \in \mathbb{R}^2_+ : \hat{V}_{\hat{\sigma}}(x,r) > (x-c)\}$ as well, since then $\hat{V}_{\hat{\sigma}}(x,r) \geq \hat{V}_\sigma(x,r) > (x-c)$. □

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