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ON FOREST ROTATION UNDER INTEREST RATE VARIABILITY

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ABSTRACT: The current literature on optimal forest rotation makes the unrealistic assumption of constant interest rate though harvesting decisions of forest stands are typically subject to relatively long time horizons. We apply the single rotation framework to extend the existing studies to cover the unexplored case of variable interest rate. We show that even in the deterministic case if the current interest rate deviates from its long-run steady state, interest rate variability may change the rotation age significantly when compared with the constant discounting case. Further, and importantly, allowing for interest rate uncertainty as a mean reverting process and forest value as a geometric Brownian motion we can provide an explicit solution for the two dimensional path-dependent optimal stopping problem. Increased interest rate volatility is shown to lengthen the optimal rotation period. Numerical calculations show that interest rate volatility has a big quantitative importance.

KEYWORDS: Optimal rotation, variable interest rates, optimal stopping, free boundary problems

JEL Subject Classification: Q23, G31, C61

1 Introduction

In forest economics the well-known model by Faustmann 1849 has been the most often used starting point in studies considering the optimal rotation period of forest stands. Under the assumption of constant timber prices, constant total cost of clear-cutting and replanting as well as constant interest rate, perfect capital markets and perfect foresight the model leads to a constant rotation period for an even aged stand, which maximizes the present value of forest stand over an infinite time horizon (see e.g. Clark 1976, Johansson and Löfgren 1985 and Samuelson 1976). The representative rotation age depends on timber price, total cost of clear-cutting and replanting, nature of forest growth as well as the interest rate.

Ongoing research has extended the basic Faustmann model under perfect foresight to allow for amenity valuation of timber (see e.g. Hartman 1976), the potential interdependence of forest stands as producers of amenity services as well as imperfect capital markets (see e.g. Koskela and Ollikainen 2001, Tahvonen and Salo and Kuuluvainen 2001). The perfect foresight assumption has been relaxed in studies focusing on the implications of stochastic timber prices (see e.g. Plantinga 1998 and Insley 2002), risk of forest fire (see e.g. Reed 1984) and/or stochastic forest growth on optimal rotation age (see e.g. Clarke and Reed 1989, 1990, Willassen 1998 and Alvarez 2001 b). The effect of uncertainties on the optimal rotation period depends on the type of uncertainty. In the case of forest fire risk modelled as a Poisson process the rotation age will become shorter due to the higher effective discount rate while in the presence of timber price and/or forest growth risk usually the reverse happens; higher risk in price or in age-dependent growth will tend to lengthen the rotation period by lowering the effective discount rate.

This rotation literature has covered several interesting cases with useful insights. There is, however, a very important issue, which has not yet been analyzed. To our knowledge in all the research associated with optimal rotation periods of forest stands the assumption of constant interest rate has been sticked to. As we know from empirical research, interest rates fluctuate over time and the implications of this empirical finding for the term structure of interest rates, asset pricing etc. have been one of the major research areas in financial economics (for an up-to-date theoretical and empirical survey in the field see Cochrane 2001, chapters 19 and 20; see also Björk 1998, chapter 17). If the investment projects would be very liquid ones, then interest rate fluctuations would not necessarily matter very much. In the case of forestry, however, the situation is different. Given the relatively slow growth rate of forests, investing in replanting is a long-term investment project, over which the expected behavior of the interest rate as the opportunity cost will matter.

In this paper we analyze the unexplored issue of what is the impact of variable interest rate on optimal forest rotation and compare the results with those obtained by using the standard, though somewhat unrealistic, assumption of constant interest rate. Since our main emphasis is to consider the impact of a variable or, more generally, a stochastic interest rate on the optimal rotation policy, we model the underlying interest rate dynamics as a parametric mean reverting process. In this way, we plan to establish robust results valid for most well-established one factor interest rate models appearing in the financial literature (cf. Björk 1998, chapter 17).

We proceed as follows: In section 2 we present a framework to study the Wicksellian single rotation problem under interest rate variability. We state a set of sufficient conditions under which the considered optimal rotation problem admits a unique solution and under which the value of the optimal policy can be obtained from an associated boundary value problem expressed as a first order linear partial differential equation subject to the standard value matching and smooth fit (or smooth pasting) conditions. From an economic point of view we show that interest rate variability will change the rotation age compared with the constant discounting case in a natural way, which depends on the relationship between the current and the long run steady state interest rate. In section 3 we illustrate our qualitative findings by using numerical computations with logistic functions modelling the interest rate in the situations where the current interest rate is below or above the long-run steady state interest rate. Having considered the deterministic case, we generalize our analysis in section 4 to study the impact of interest rate uncertainty on optimal forest rotation. We specify interest rate uncertainty as a mean reverting process an forest value as a geometric Brownian motion and give an explicit solution for the associated two dimensional optimal stopping problem. We also establish that interest rate uncertainty prolongs the rotation period and provide numerical illustration about its significant quantitative role. Section 5 presents some concluding remarks.

2 The Wicksellian Rotation Problem with Variable Interest Rate

In this section we formulate the Wicksellian rotation problem in more general terms than usually by allowing interest rate variability and study the relationship between the rotation periods under variable and constant discounting.

The underlying dynamics for the forest value X_t and interest rate r_t are described as

$$X_t' = \mu(X_t), \quad X_0 = x \tag{2.1}$$

and

$$r_t' = \alpha(r_t), \quad r_0 = r \tag{2.2}$$

where the mappings $\mu : \mathbb{R}_+ \mapsto \mathbb{R}$ and $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}$ are assumed to be continuously differentiable with Lipschitz-continuous derivative on $\mathbb{R}_+$. In order to capture the economically plausible models of the optimal rotation problem, we assume that there is a $\hat{r} > 0$ such that $\alpha(r) \gtreqless 0$, when $r \lesseqgtr \hat{r}$, that $\lim_{r\downarrow 0} \alpha(r) = 0$, that $\lim_{x\downarrow 0} \mu(x) = 0$ and that there is a $\hat{x} > 0$ such that $\mu(x) \gtreqless 0$, when $x \lesseqgtr \hat{x}$. In other words, we assume that the origin is an unstable equilibrium point for the two dimensional process (X_t, r_t) and that (X_t, r_t) tends towards the asymptotically stable long run steady state $(\hat{x}, \hat{r})$ for any possible interior initial state $(x, r) \in \mathbb{R}^2_+$. This lies in conformity with mean reverting interest rate modelling developed, and provided empirical support, in financial economics. However, from an economical point of view only the non-negative interest rates are of interest and, therefore, we shall concentrate on that case. As usually, we denote as

$$\mathcal{A} = \mu(x)\frac{\partial}{\partial x} + \alpha(r)\frac{\partial}{\partial r} \tag{2.3}$$

the differential operator associated with the intertemporally time-homogeneous process (X_t, r_t).

Given the underlying dynamics, we consider the *Wicksellian single rotation problem*

$$V(x,r) = \sup_{t\geq 0}\left[e^{-\int_0^t r_s ds} g(X_t)\right], \tag{2.4}$$

where $g : \mathbb{R}_+ \mapsto \mathbb{R}$ is a twice continuously differentiable, non-decreasing, and concave mapping (i.e. $g \in C^2(\mathbb{R}_+)$, $g'(x) \geq 0$, and $g''(x) \leq 0$ for all $x \in \mathbb{R}_+$) denoting the payoff accrued from exercising the irreversible harvesting opportunity and satisfying the boundary condition $g(\hat{x}) > 0$ (implying that $g(x) \geq g(\hat{x}) > 0$ for all $x > \hat{x}$). It is now a simple exercise in ordinary analysis to demonstrate that given our smoothness assumptions, the optimal rotation problem (2.4) can be restated as (see, for example, Øksendal 1998, p. 199)

$$V(x,r) = g(x) + F(x,r), \tag{2.5}$$

where the integral term

$$F(x,r) = \sup_{t\geq 0}\int_0^t e^{-\int_0^s r_y dy}\left[g'(X_s)\mu(X_s) - r_s g(X_s)\right] ds$$

constitutes the early exercise premium accrued from undertaking optimally the irreversible policy prior expiration. We can present an auxiliary verification lemma which can be applied for solving either the optimal stopping problem (2.4) or its equivalence (2.5) (see Alvarez and Koskela 2001). This states a set of sufficient conditions (in terms of variational inequalities) which can be applied for deriving a majorant for the value of the optimal rotation problem (2.4) and also establishes a set of sufficient conditions which can be applied for deriving the explicit form of the early exercise premium $F(x,r)$. As usually in optimal stopping theory, we denote the continuation region (i.e. the waiting or do-nothing region) where exercising the harvesting opportunity is suboptimal as $C = \{(x,r) \in \mathbb{R}^2_+ : V(x,r) > g(x)\}$ and the stopping region (i.e. the immediate exercise region, cutting region) as $\Gamma = \{(x,r) \in \mathbb{R}^2_+ : V(x,r) = g(x)\}$. Clearly, the set $g^{-1}(\mathbb{R}_-) = \{x \in \mathbb{R}_+ : g(x) < 0\}$ is a subset of the continuation region C since the decision maker can always attain at least a non-negative payoff by waiting up to the first moment when X_t arrives to the set where the payoff $g(x)$ is positive. We assume now that $(x,r) \in \bar{C} = \{(x,r) \in \mathbb{R}^2_+ : g'(x)\mu(x) > rg(x)\}$ and define the stopping date $\tau = \inf\{t \geq 0 : (X_t, r_t) \notin \bar{C}\}$, which give

$$V(x,r) \;\;\geq\;\; e^{-\int_0^\tau r_s ds} g(X_\tau) = g(x) + \int_0^\tau e^{-\int_0^s r_y dy}\left[g'(X_s)\mu(X_s) - r_s g(X_s)\right] ds > g(x)$$

implying that $\bar{C} \subseteq C$. Equation (2.5) implies that if there is a finite date $t^* \in \mathbb{R}_+$ at which the opportunity is exercised, then we necessarily have that $g'(X_{t^*})\mu(X_{t^*}) = r_{t^*} g(X_{t^*})$. On the other hand, t^* can be a maximum only if also the second order local sufficiency condition $\mu(X_{t^*})[g''(X_{t^*})\mu(X_{t^*}) + g'(X_{t^*})(\mu'(X_{t^*}) - r_{t^*})] < \alpha(r_{t^*})g(X_{t^*})$ is met. Consequently, we define implicitly the boundary curve of the continuation region as $\{(x,r) \in \mathbb{R}^2_+ : g'(x)\mu(x) = rg(x)\}$. Implicit differentiation then yields that along the boundary we have

$$\frac{dx}{dr} = \frac{g''(x)\mu(x) - g'(x)(r - \mu'(x))}{g(x)}$$

provided that $g(x) > 0$. This means that if the payoff $g(x)$ is non-decreasing and concave and $r > \mu'(x)$, then the boundary at which rotation is optimal is a decreasing function of the current rate of interest. Our main result characterizing the optimal rotation policy and its value for this broad class of problems is now summarized in the following

Theorem 2.1. *Assume that* $D = (\hat{x}, \infty) \times (\hat{r}, \infty) = \{(x,r) \in \mathbb{R}^2_+ : \mu(x) < 0, \alpha(r) < 0\} \subset \{(x,r) \in \mathbb{R}^2_+ : g'(x)\mu(x) < rg(x)\}$ *and that* $\mu(x)[g''(x)\mu(x) - g'(x)(r - \mu'(x))] - \alpha(r)g(x) < 0$ *for all* $(x,r) \in \mathbb{R}^2_+ \backslash D$. *Then, for any* $(x,r) \in C$ *the optimal rotation date* $t^* = \inf\{t \geq 0 : (X_t, r_t) \notin C\} < \infty$ *is the root of the equation*

$$g'(X_{t^*})\mu(X_{t^*}) = r_{t^*} g(X_{t^*}).$$

Moreover, the value satisfies the boundary value problem

$$\begin{aligned} &\mu(x)\frac{\partial V}{\partial x}(x,r) + \alpha(r)\frac{\partial V}{\partial r}(x,r) - rV(x,r) = 0, \quad (x,r) \in C \\ &V(x,r) = g(x), \quad \frac{\partial V}{\partial x}(x,r) = g'(x), \quad \frac{\partial V}{\partial r}(x,r) = 0, (x,r) \in \partial C. \end{aligned} \tag{2.6}$$

Proof. See Alvarez and Koskela 2001. □

Theorem 2.1 provides a set of sufficient conditions under which the optimal rotation problem admits a unique solution and the value of optimal policy can be determined from an associated boundary value problem. Figure 1 describes the phase diagram of

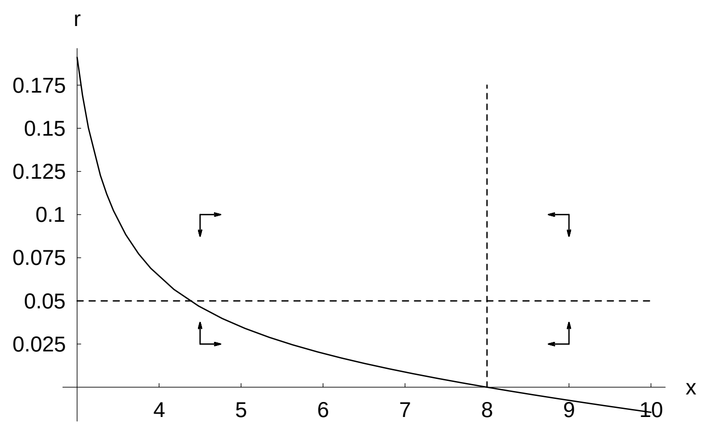

Figure 1: The Phase Diagram of the Controlled System

the controlled system under the assumptions of our Theorem 2.1. As was required in

Theorem 2.1 the equilibrium state $(\hat{x}, \hat{r})$ of the controlled system has to be in the set where the present value of the forest is decreasing over time, i.e., above the boundary of the continuation region C. If that condition is met, then the two-dimensional system (X_t, r_t) will tend towards the stopping region from any initial state in the do-nothing region C and hit its boundary in finite time.

Given that new characterization we next explore the relationship between the rotation periods with variable and constant discounting under the plausible assumption that there exists the long-run steady state interest rate in the deterministic case. The answer is summarized in the following

Theorem 2.2. *If the conditions of Theorem 2.1 are met, then the rotation period in the presence of variable discounting is shorter (longer) than the rotation period in the presence of constant discounting if* $r < \hat{r}$ $(r > \hat{r})$.

Proof. It is clear that in the absence of interest rate variability (i.e. when $r_t' = 0$ for all $t \geq 0$ and $r_0 = r$) the optimal rotation period $\tilde{t}$ satisfies the optimality condition $g'(X_{\tilde{t}})\mu(X_{\tilde{t}}) = rg(X_{\tilde{t}})$. Consequently, we find that

$$f(\tilde{t}) = g'(X_{\tilde{t}})\mu(X_{\tilde{t}}) - r_{\tilde{t}}g(X_{\tilde{t}}) = (r - r_{\tilde{t}})g(X_{\tilde{t}}) \gtreqless 0 \quad r \gtreqless \hat{r},$$

since

$$r_t \gtreqless r \quad \forall t \geq 0 \quad \text{whenever} \quad r \lesseqgtr \hat{r}.$$

Therefore, we find that $\tilde{t} \gtreqless t^*$ whenever $r \lesseqgtr \hat{r}$. □

According to Theorem 2.2 the rotation period with variable discounting falls short of the one with constant discounting when the current interest rate is known with certainty to increase over time. This is natural because in that case the opportunity costs of not harvesting increases over time. The reverse happens in the case of falling interest rate when the opportunity cost of not harvesting goes down.

3 A Numerical Illustration in the Case of Logistic Growth

According to Theorem 2.2, the rotation period with variable interest rate is shorter (longer) that the one in the presence of a constant interest rate when the current interest rate is smaller (higher) than its long run steady state value. Now we illustrate this finding quantitatively by using a model based on logistic (or mean-reverting) dynamics of our two dimensional process (X_t, r_t). We now assume that

$$X_t' = \mu X_t(1 - \gamma X_t), \quad X_0 = x \tag{3.1}$$

and

$$r_t' = \alpha r_t(1 - \beta r_t), \quad r_0 = r, \tag{3.2}$$

where μ, γ, α, and β are exogenously determined non-negative constants. It is now a simple exercise in ordinary analysis to demonstrate that

$$X_t = \frac{xe^{\mu t}}{1 + \gamma x(e^{\mu t} - 1)} \quad \text{and} \quad r_t = \frac{re^{\alpha t}}{1 + \beta r(e^{\alpha t} - 1)}.$$

Moreover, standard differentiation yields $d\ln r_t = (\alpha - \alpha\beta r_t)dt$ implying that

$$\ln(r_t/r) = \alpha t - \int_0^t \alpha\beta r_s ds$$

and, therefore, that

$$e^{-\int_0^t r_s ds} = \left(1 + \beta r(e^{\alpha t} - 1)\right)^{-1/(\alpha\beta)}.$$

Consequently, if the payoff is described by $g(x) = x - c$, then (2.4) reads as

$$V(x,r) = \sup_{t\geq 0}\left[\left(1 + \beta r(e^{\alpha t} - 1)\right)^{-1/(\alpha\beta)}\left(\frac{xe^{\mu t}}{1 + \gamma x(e^{\mu t} - 1)} - c\right)\right]$$

Therefore if an optimal rotation date t^* exists, it is implicitly given by the equation

$$\frac{\mu x(1-\gamma x)e^{\mu t^*}}{(1 + \gamma x(e^{\mu t^*} - 1))^2} = \left(\frac{xe^{\mu t^*}}{1 + \gamma x(e^{\mu t^*} - 1)} - c\right)\frac{re^{\alpha t^*}}{1 + \beta r(e^{\alpha t^*} - 1)}.$$

The optimal exercise date t^* is illustrated in Figure 2 under the assumptions that $g(x) = x - c$, $c = 7, \mu = 1\%, \gamma = 2\%, \alpha = 1\%, x = 5$, and $\beta = 100/3$ (implying that the long run steady state can be defined as $(\hat{x}, \hat{r}) = (50, 3\%)$). Figure 2 illustrates the case when the current interest rate r is below its long-run steady state value $\hat{r}$. The solid line describes the optimal rotation period in the presence of a variable interest rate, while the dotted line the optimal rotation period with constant discounting. The rotation period with variable discounting falls short of the rotation period with constant discounting because the interest rate is known with certainty to increase over time. One can see that the difference between the rotation periods becomes larger with lower interest rates. This is obvious because in the presence of smaller current interest rate it will increase much more over time with variable discounting. Therefore, the optimal rotation periods will differ more in that case.

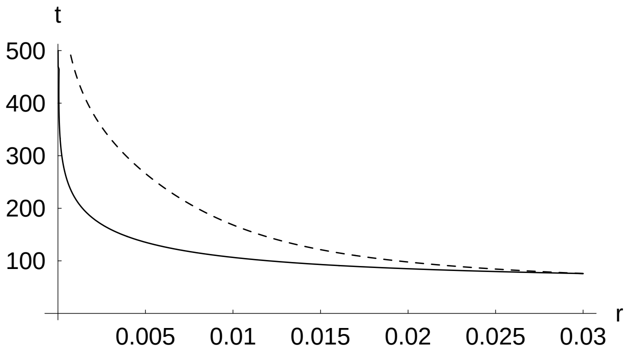

Figure 2: The optimal exercise dates as function of the current interest rate r when $r < \hat{r}$

Analogously, Figure 3 illustrates the alternative case where the current interest rate r is above its long-rung steady sate value $\hat{r}$. Again, the solid line describes the optimal rotation period in the presence of variable interest rate, while the dotted line describes the optimal rotation period with constant discounting. Naturally, contrary to the findings in Figure 2, the rotation period with variable discounting exceeds the one with constant discounting because the interest rate is known to decrease over time. Compared with Figure 2 now the reverse happens also in the sense that the difference between the optimal rotation periods with and without interest rate variability becomes larger with higher interest rates. The reason for that is that in the presence of higher current interest rate it will decrease much more over time with variable discounting.

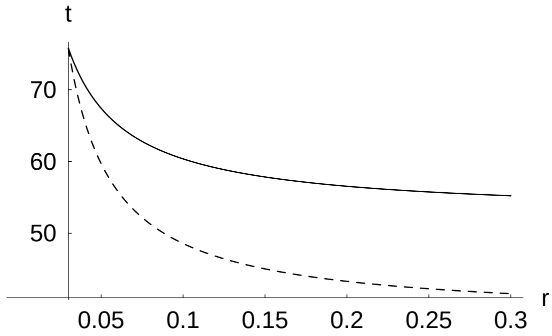

Figure 3: The optimal exercise dates as function of the current interest rate r when $r > \hat{r}$

4 Interest Rate Uncertainty and Forest Rotation: A Solvable Model

In the analyzes we have carried out thus far, the underlying dynamics for the forest value X_t and the interest rate r_t have been deterministic. This is because we first wanted to consider the impact of variable discounting on the optimal rotation period in the simpler case. Of course, in light of the length of the standard forest rotation decisions the assumption of completely deterministic dynamics is difficult to defend, to say the least. To mention a specific example: do we know the behavior of the interest rates over the next five decades? Certainly the right answer is: We do not know, but we still might have a good knowledge about the stochastic process generating the interest rate fluctuations. In what follows, we model the stochastic interest rate dynamics as an explicitly parametrized mean reverting process (which lies in conformity with empirical evidence, see e.g. Cochrane, 2001, chapters 19 and 20) and forest value in a simpler way.

Consider the following (path-dependent) optimal rotation problem

$$V(x,r) = \sup_{\tau} \mathbf{E}_{(x,r)} \left[e^{-\int_0^\tau r_s ds} X_\tau \right], \tag{4.1}$$

where the underlying processes (X_t, r_t) evolve according to the dynamics described by the following stochastic differential equations

$$dr_t = (a - br_t)dt + cdW_t, \quad r_0 = r \tag{4.2}$$

and

$$dX_t = \mu X_t dt + \sigma X_t d\hat{W}_t, \quad X_0 = x, \tag{4.3}$$

where $a, b, c, \sigma, \mu \in \mathbb{R}_+$ are known exogenously given constants and W_t and $\hat{W}_t$ are potentially correlated Wiener processes (under the objective probability measure $\mathbb{P}$) with a known correlation coefficient $\rho \in [-1, 1]$. It is worth emphasizing that the interest rate model (4.2) is known in financial economics as the Vasiček-model of the interest rate. This model has the natural property of being mean reverting in the sense that it will tend to revert to the mean level a/b.

Having characterized the underlying stochastic dynamics in (4.2) and (4.3) and the corresponding optimal single rotation problem (4.1) we can now state the following

Lemma 4.1. *The path-dependent optimal rotation problem (4.1) can be re-expressed as an ordinary path-independent optimal stopping problem*

$$V(x, r) = xe^{-r/b} \sup_{\tau} \mathbf{E}_r \left[e^{-\theta\tau + \hat{r}_\tau / b} \right], \tag{4.4}$$

where

$$d\hat{r}_t = \left(a - \frac{c^2}{b} + \rho c\sigma - b\hat{r}_t \right) dt + c dW_t, \quad \hat{r}_0 = r \tag{4.5}$$

and

$$\theta = \frac{a}{b} + \frac{\rho c \sigma}{b} - \mu - \frac{c^2}{2b^2}$$

can be interpreted as a "risk-adjusted" discount rate.

Proof. See Appendix A. □

Lemma 4.1 is important in the sense that under the assumptions we have made concerning the stochastic processes modelling the interest rate and the forest value, the path dependent single rotation problem can actually be transformed into an ordinary path-independent optimal stopping problem. Our main new result in this section is now summarized in the following

Theorem 4.2. *Assume that the risk-adjusted discount rate is positive (i.e. $\theta > 0$). Then the value of the single rotation problem (4.1) reads as*

$$V(x, r) = xe^{-r/b} \psi(r) \sup_{y \geq r} \left[\frac{e^{y/b}}{\psi(y)} \right] = \begin{cases} x, & r \geq r^* \\ xe^{(r^* - r)/b} \frac{\psi(r)}{\psi(r^*)}, & r < r^* \end{cases}$$

where

$$\psi(r) = \int_0^\infty t^{\theta/b - 1} e^{\sqrt{\gamma}(br - \kappa)t - \frac{1}{2}t^2} dt,$$

$\kappa = a - c^2/b + \rho c\sigma$ and $\gamma = 2/(c^2 b)$. Moreover the optimal exercise threshold r^* is the unique root of the ordinary first order condition $\psi(r^*) = b\psi'(r^*)$. Especially, $r^* > \mu$ for all $c > 0$ and $r^* = \mu$ when $c = 0$.

Proof. See Appendix B. □

Theorem 4.2 demonstrates that the path-dependent optimal rotation problem (4.4) is explicitly solvable whenever the absence of speculative bubbles condition $\theta > 0$ is satisfied. It is worth noticing that in the absence of uncertainty the condition $\theta > 0$ can be simply expressed as $a/b > \mu$ meaning that the steady state interest rate a/b exceeds the growth rate μ of the forest value. On the other hand, in the presence of uncertainty the absence of speculative bubbles condition $\theta > 0$ can also be re-expressed as

$$\frac{a}{b} > \mu + \frac{c^2}{2b^2} - \frac{\rho c\sigma}{b}$$

Thus, we find that the condition $\theta > 0$ is strengthened by the presence of uncertainty whenever the correlation between the two driving Brownian motions, ρ, is non positive and is weakened whenever the correlation is positive. Moreover, and importantly, higher volatility increases the required exercise premium and thus prolongs the rotation period. It is also worth noticing that Theorem 4.2 demonstrates that the waiting premium reads for all $r < r^*$ as

$$V(x,r) - x = \left(e^{(r^*-r)/b}\frac{\psi(r)}{\psi(r^*)} - 1\right)x,$$

which is linearly dependent on the current value of the forest stand. Thus, we find that required exercise premium is independent of the underlying value process X_t but still dependent on the parameters of that process (through the risk-adjusted discount factor θ and the mean reversion parameter κ).

Remark: It is worth noticing that since

$$dX_t^\beta = \left(\beta\mu + \frac{1}{2}\beta(\beta-1)\sigma^2\right)X_t^\beta dt + \beta\sigma X_t^\beta d\hat{W}_t,$$

the result of Theorem 4.2 can be applied for solving the associated optimal stopping problem

$$H(x,r) = \sup_\tau \mathbf{E}_{(x,r)}\left[e^{-\int_0^\tau r_s ds}X_\tau^\beta\right], \tag{4.6}$$

where $\beta \in \mathbb{R}$ is a known parameter measuring the curvature of the mapping x^β. As is clear from Theorem 4.2, in that case the value of the stopping problem (4.6) reads as

$$H(x,r) = x^\beta e^{-r/b}\tilde{\psi}(r)\sup_{y\geq r}\left[\frac{e^{y/b}}{\tilde{\psi}(y)}\right] = \begin{cases} x^\beta, & r \geq \tilde{r} \\ x^\beta e^{(\tilde{r}-r)/b}\frac{\tilde{\psi}(r)}{\tilde{\psi}(\tilde{r})}, & r < \tilde{r} \end{cases}$$

where

$$\tilde{\psi}(r) = \int_0^\infty t^{\tilde{\theta}/b-1}e^{\sqrt{\gamma}(br-\tilde{\kappa})t-\frac{1}{2}t^2}dt,$$

$\tilde{\kappa} = a - c^2/b + \rho c\sigma\beta$ and

$$\tilde{\theta} = \frac{a}{b} + \frac{\rho c\beta\sigma}{b} - \beta\mu - \frac{1}{2}\beta(\beta-1)\sigma^2 - \frac{c^2}{2b^2}.$$

Moreover, the optimal exercise threshold $\tilde{r}$ is the unique root of the ordinary first order condition $\tilde{\psi}(\tilde{r}) = b\tilde{\psi}'(\tilde{r})$.

Finally, we characterize the quantitative significance of the volatility coefficient c by numerical illustrations. Assume that $b = 0.1$, $a = 0.045b$ (implying the expected steady state interest rate $a/b = 4.5\%$), $\mu = 1\%$, $\rho = 0.5$, and $\sigma = 0.05$ (implying that the critical volatility above which absence of speculative bubbles condition is violated is $c \approx 2.91\%$). Then, the optimal threshold r^* and required exercise premium $r^* - \mu$ as a function of the underlying volatility coefficient is

c	0.5%	1%	2%	2.5%
r^*	1.32%	2.13%	4.99%	7.73%
$r^* - \mu$	0.32%	1.13%	3.99%	6.73%

Thus, we find that the required exercise premium increases from 0.32% to 6.73% as volatility goes up from 0.5% to 2.5%. In order to illustrate our results in the negative correlation case, we assume that $b = 0.1$, $a = 0.045b$, $\mu = 1\%$, $\rho = -0.5$, and $\sigma = 0.05$ (implying that the critical volatility above which absence of speculative bubbles condition is violated is $c \approx 2.41\%$). In this case the optimal threshold and required exercise premium as a function of the underlying volatility coefficient are

c	0.5%	1%	2%	2.4%
r^*	1.34%	2.26%	6.24%	14.92%
$r^* - \mu$	0.34%	1.26%	5.24%	13.92%

Thus, we find that the required exercise premium increases from 0.34% to 13.92% as volatility increases from 0.5% to 2.4%. To summarize, according to these numerical illustrations, higher interest rate volatility has a big effect on the required exercise premium, thus meaning that higher volatility lengthens the rotation period strongly.

5 Conclusions

There is currently an extensive literature about the determination of optimal forest rotation under various circumstances when amenity valuation of forest stands matters, when capital markets are imperfect so that landowners might be subject to credit rationing or when there is uncertainty about timber prices and/or forest growth due either to forest growth uncertainty or to risk of forest fire. Undoubtedly this literature has provided useful insights about the potential determinants of forest rotation. There is, however, an important issue, which has not yet been analyzed. To our knowledge all the literature makes a simplifying but in the forestry case an unrealistic assumption that the interest rate is constant. Clearly the irreversible harvesting decision of forest stands is a decision subject to a relatively long time horizon. Hence, given the relatively slow growth rate of forests, thinking about harvesting and investing in replanting is a long-term investment project over which the behavior of interest rates as the opportunity cost should matter a lot.

In this paper we have used the Wicksellian single rotation framework to extend the existing studies to cover the unexplored case of variable interest rate. From an economic point of view we have established several new findings. First, the variability of interest rate will change the rotation age compared with the constant discounting in a way which depends on the relationship between the current and long run steady state interest rate. If the current interest rate is lower than the asymptotically stable one, then the variable interest rate rotation age is lower than the one with constant discounting. The reverse happens in the case when current interest rate is above the long-run steady state interest rate. We illustrated this qualitative finding also quantitatively by using numerical computations in section 3 with logistic functions. Second, we have demonstrated that allowing for interest rate uncertainty a higher volatility will increase the optimal rotation period. We used a parametric example by modelling the interest rate as a mean-reverting process by using the Vasičeck model of the interest rate and the forest value as a geometric Brownian motion to get an explicit solution for a two-dimensional path-dependent optimal stopping problem. We also provide a numerical illustration which shows a significant quantitative importance of interest rate volatility for the optimal rotation age. Whether our conclusions remain valid in the Faustmannian ongoing rotation framework is an open issue beyond the scope of this paper. This is a challenging problem left for future research.

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A Proof of Lemma 4.1

Since

$$X_t = x \exp((\mu - \beta^2/2)t + \beta \hat{W}_t)$$

we find by applying Itô's theorem to the mapping $r \mapsto r/b$ that

$$\frac{r_t - r}{b} = \frac{at}{b} - \int_0^t r_s ds + \frac{c}{b} W_t,$$

which in turn implies that the discount factor can be re-expressed as

$$e^{-\int_0^t r_s ds} = e^{(r_t - r - at - cW_t)/b}.$$

Therefore, we find that the present value of the forest can be expressed as

$$e^{-\int_0^t r_s ds} X_t = x e^{(r_t - r)/b - \theta t} M_t,$$

where

$$M_t = e^{\sigma \hat{W}_t - cW_t/b - \frac{1}{2}(\sigma^2 + c^2/b^2 - 2\sigma c \rho/b)t}$$

is a positive exponential $\mathcal{F}_t$-martingale. Consequently, we find that the path-dependent Wicksellian optimal rotation problem (4.1) can be re-expressed as an ordinary path-independent optimal stopping problem

$$V(x,r) = xe^{-r/b} \sup_{\tau} \mathbf{E}_{(x,r)} \left[e^{-\theta\tau + r_\tau/b} M_\tau \right], \tag{A.1}$$

where

$$\theta = \frac{a}{b} + \frac{\rho c \sigma}{b} - \mu - \frac{c^2}{2b^2}$$

can be interpreted as a "risk-adjusted" discount rate. Defining the equivalent measure $\mathbb{Q}$ as $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_t$ we can now re-express (A.1) as

$$V(x,r) = xe^{-r/b} \sup_{\tau} \mathbf{E}^{\mathbb{Q}}_{(x,r)} \left[e^{-\theta\tau + r_\tau/b} \right], \tag{A.2}$$

where the interest rate process r_t evolves under $\mathbb{Q}$ according to the dynamics described by the stochastic differential equation

$$dr_t = \left(a - \frac{c^2}{b} + \rho c b - b r_t \right) dt + \sigma d\tilde{W}_t, \quad r_0 = r$$

where $\tilde{W}_t$ is a standard Brownian motion under the equivalent measure $\mathbb{Q}$. However, given the strong uniqueness of a solution for the stochastic differential equation above we finally find that the rotation problem (4.1) can be rewritten in the path-independent form (4.4) defined under the objective measure $\mathbb{P}$.

B Proof of Theorem 4.2

Proof. Since

$$L(r) = \sup_{\tau} \mathbf{E}_r \left[e^{-\theta\tau + \hat{r}_\tau/b} \right]$$

is an ordinary path independent optimal stopping problem of a linear diffusion and, therefore, can be solved by relying on ordinary variational inequalities, the alleged result is a direct implication of Theorem 3 in Alvarez 2001 a. It is, therefore, sufficient to determine the increasing fundamental solution of the ordinary second order differential equation

$$\frac{1}{2}\sigma^2 u''(r) + (\kappa - br)u'(r) - \theta u(r) = 0,$$

where $\kappa = a - c^2/b + \rho cb$. Making the transformation $u(r) = v(z(\kappa - br))$,where $z \in \mathbb{R}$ is an unknown constant, and defining the variable $y = z(\kappa - br)$ then yields that

$$v''(y) - \frac{2}{\sigma^2 b z^2} y v'(y) - \frac{2\theta}{\sigma^2 b^2 z^2} v(y) = 0.$$

Choosing $z^2 = 2/(b\sigma^2)$, then finally implies that the differential equation can equivalently be expressed as

$$v''(y) - yv'(y) - \frac{\theta}{b} v(y) = 0.$$

As is demonstrated in Taylor (1968), and Borodin and Salminen (2002), pp. 136–137, the decreasing fundamental solution of this equation reads as (this is needed since $z(a - br)$ is decreasing)

$$v(y) = e^{\frac{y^2}{4}} D_{-\theta/b}(y),$$

where

$$D_{-\theta/b}(z) = \frac{1}{\Gamma(\theta/b)} e^{-\frac{z^2}{4}} \int_0^\infty t^{\theta/b-1} e^{-zt-\frac{1}{2}t^2} dt$$

denotes the parabolic cylinder function of order $-\theta/b$. Inserting then $y = \sqrt{2/(b\sigma^2)}(a - br)$ yields the alleged result and completes the proof of our Theorem. $\square$

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