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Keskusteluaiheita – Discussion papers

No. 789

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SADDLES, INDETERMINACY AND BIFURCATIONS IN AN OVERLAPPING GENERATIONS ECONOMY WITH A RENEWABLE RESOURCE****

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**** Earlier versions of the paper have been presented at the joint theory/macro workshop in the University of Pennsylvania, the 2001 conference of EARE in Southampton, the 2001 meeting of the Society for Economic Dynamics in Stockholm. We would like to thank the participants as well as Roger Farmer for useful comments. We are also indebted to Dr. Sakari Uimonen for assistance in numerical computations. Koskela acknowledges financial support from the Academy of Finland, and thanks Research Department of the Bank of Finland and CESifo in the University of Munich for their hospitality.

ISSN 0781-6847 18.02.2002

KOSKELA, Erkki – **OLLIKAINEN**, Markku – **PUHAKKA**, Mikko, **SADDLES, INDETERMINACY AND BIFURCATIONS IN AN OVERLAPPING GENERATIONS ECONOMY WITH A RENEWABLE RESOURCE.** Helsinki: ETLA, Elinkeinoelämän Tutkimuslaitos, The Research Institute of the Finnish Economy, 2002,30 p. (Keskusteluaiheita, Discussion Papers, ISSN, 0781-6847; no. 789).

ABSTRACT: We incorporate a renewable resource into an overlapping generations model with standard, well-behaved utility and constant returns production functions. Besides being a factor of production the resource serves as a store of value. We characterize dynamics, efficiency and stability of steady state equilibria, and show how their nature in the presence of 'well-behaved' resource stock growth depends on the size of the intertemporal elasticity of substitution in consumption. If that elasticity is at least half, but not exactly one, steady states are saddle points. For the values of elasticity less that one half we use a parametric example to demonstrate the existence of stable equilibria (indeterminacy) and a subcritical flip bifurcation. These findings lie in conformity with empirics, which suggest that exploited fish populations exhibit not only the conventionally analyzed steady yields, but also cycles and irregularities.

Keywords: overlapping generations, renewable resources, bifurcations.

JEL classification: D90, Q20, C62.

1. INTRODUCTION

The overlapping generations framework is a rich source for many kinds of interesting dynamics. Endogenous cycles, chaos, bifurcations, indeterminacy, and sunspot equilibria can be observed in these models.[1] The overlapping generations structure in itself is not necessarily a source for indeterminacy and other types of "nonstandard dynamics". For example, Diamond's (1965) overlapping generations model with production, but without government debt, and with a regular neoclassical production function and a saving function with a positive interest rate elasticity exhibits determinate dynamics.

Since Gale (1973) it has been known that indeterminacy and cycles are possible in the overlapping generations models. By applying the theory of flip bifurcations Grandmont (1985) showed in a simple monetary model without capital accumulation that periodic equilibria and chaos can occur, when the consumer's offer curve is backward bending. To generate backward bending offer curves, it is necessary to have the intertemporal elasticity of substitution smaller than one.[2] The small size of this elasticity seems to be essential to obtain cycles in one-dimensional dynamics. Indeed, it is also the case that the basic overlapping generations model with production can have indeterminate equilibria, if saving is a decreasing function of the interest rate.[3]

Reichlin (1986, 1990) has shown that endogenous cycles, however, can be observed in models with elastic labor supply or a two-sector technology producing a consumption and a capital good, but without the restriction on the intertemporal elasticity of substitution. Farmer (1986) shows in an overlapping generations model with production and the government as a lender that a bifurcation may emerge giving rise to invariant cycles.

Calvo (1978) showed that indeterminate equilibria can occur in overlapping generations models.[4] His examples were not restricted to a monetary model. A two-sector model without a nominal asset can exhibit indeterminacy.

An overlapping generations model with production and government debt has two steady states, one of which (with zero level of debt) is locally stable (indeterminate).[5] More recent examples of this class of models are Schreft and Smith (1997) and Boyd and Smith (1998), who introduce market imperfections to a monetary growth model, and obtain multiple steady states and cycles. The fact that market imperfections cause complex dynamics is not surprising, since they are important sources for indeterminacy in many other models, too.[6]

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1 Models with indeterminate equilibria are prime candidates for having sunspot equilibria. Boldrin and Woodford (1990) have an extensive survey on cycles and chaos in overlapping generations models. For a survey on sunspot equilibria, see Cass and Shell (1989).

2 Grandmont demonstrated the existence of cycles when the Arrow-Pratt relative risk aversion of the old agents exceeds two, in which case saving is a decreasing function of the interest rate. Jullien (1988) has shown that Grandmont's result can be generalized to a monetary economy with production if there is "enough" substitutability between factors of production. Grandmont (1985) is based on backward dynamics. For forward dynamics, see the model by Benhabib and Day (1982) which also exhibits periodic equilibria.

3 For a demonstration, see e.g. Azariadis (1993).

4 Model by Grandmont with backward bending offer curve naturally has a lot of indeterminate equilibria.

5 For a demonstration see Azariadis (1993), p. 199 and 201.

6 See the survey by Benhabib and Farmer (1999).

In the renewable resource literature Kemp and Long (1979) and Mourmouras (1991, 1993) have analyzed the sustainable use of renewable resources within the overlapping generations framework. They demonstrate the generally well-known fact that competitive equilibria in overlapping generations economies may be inefficient, but do not study dynamics and stability of the model. Koskela, Ollikainen and Puhakka (2001) analyze the dynamics of an overlapping generations renewable resource economy with quasi-linear preferences, and show that all equilibria, be they efficient or not, are saddles. Our present paper, however, utilizes a more general utility function, and yields strikingly different results. Finally, Olson and Knapp (1997) study an overlapping generations economy with an exhaustible resource. Among other things they demonstrate the existence of cycles and multiple equilibria.

In this paper we study the dynamical properties of an overlapping generations model with a renewable natural resource. The empirical relevance of this research issue is easy to establish. It is well known that exploited renewable populations may behave in many different ways. By using historical data Caddy and Gulland (1983) have found that while some fish stocks provide steady yields, some others provide cyclical, irregular or spasmodic yields.[7] Hilborn and Walters (1992) stress that, in fact, most species do not appear to be capable of producing steady yields, and they wonder why studies of fisheries management have not devoted attention to this feature, which has potentially significant economic implications. Our paper focuses on the possibility that a renewable resource economy may have other type of dynamics than a conventional steady state.

In our model the resource with a strictly concave growth function serves as a factor of production and a store of value. A model with a renewable resource differs from a standard overlapping generations model in one important aspect, the appearance of the "well-behaved" biological growth function. This means that the gross return from investing in the resource is not a linear function of the resource stock. The strict concavity of the growth function does not bring about any non-convexity into the model. Yet, in addition to saddles we are able to show that our model can possess indeterminate equilibria, and flip bifurcations.

Our model will be a two dimensional planar system. We characterize the steady state equilibrium, compare competitive and efficient solutions, and in particular, study its stability properties. The nature of the steady state equilibrium will depend on the value of the intertemporal elasticity of substitution in consumption. In particular, if the size of this elasticity is at least half, but different from one, then steady states are saddle points. This result holds for a general strictly concave resource growth function.

For the values of intertemporal elasticity of substitution less than one half we use a parametric example with logistic resource growth to demonstrate the existence of a subcritical flip bifurcation in the case of an inefficient equilibrium. This means that a repelling two-cycle emerges on that side of the flip bifurcation, where the steady state is stable. This indeterminacy indicates a possibility for sunspot equilibria. When the intertemporal elasticity of substitution equals unity, the dynamical system reduces to a first-order nonlinear differ-

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7 The meaning of steady state, cyclical and irregular is obvious; spasmodic means fish stocks, which have produced high yields and then collapsed without any major recovery. Examples of these cases are North Sea turbot fishery, Baleares hake fishery, Norwegian fishery for juvenile herring and Pacific sardine fisheries, respectively (see Hilborn and Walters 1992, Ch. 2).

ence equation in the resource stock and the system is stable. In our model flip bifurcations and complex dynamics emerge due to the mixture of low elasticity of substitution in consumption and “well-behaved” logistic growth.

We proceed as follows. The elements of an overlapping generations economy with a renewable resource are presented, and the equilibrium conditions of the economy characterized in section 2. Conditions for a unique steady state and its efficiency are analyzed in section 3. In section 4 we study dynamical equilibria of a planar system for harvesting and resource stock, and characterize the case where all the stationary equilibria are saddle points. Since saddle point equilibria may not exist if the intertemporal elasticity of substitution in consumption is low enough, section 5 studies this case. Finally, section 6 summarizes our findings.

2. THE MODEL AND THE EQUILIBRIUM CONDITIONS

We consider an overlapping generations economy where agents live for two periods. There is no population growth. Agents maximize the following intertemporally additive lifetime utility function

(1) $$V = u(c_1^t) + \beta u(c_2^t),$$

where c_i^t denotes the period i (=1,2) consumption of consumer-worker born at time t and $\beta = (1+\delta)^{-1}$ with δ being the rate of time preference. We assume that $u' > 0$, $u'' < 0$ and the Inada conditions, i.e. $\lim_{c\to\infty} u'(c) = 0$ and $\lim_{c\to 0} u'(c) = \infty$. The young are endowed with one unit of labor, which they supply inelastically to firms in the consumption goods sector. Labor earns a competitive wage. The representative consumer-worker uses the wage to buy the consumption good and to save. He can save in the financial asset or buy the available stock of the renewable resource.

The firms in the consumption goods sector have a constant returns to scale technology, $F(H_t, L_t)$, to transform the harvested resource (H_t) and labor (L_t) into output. This technology can be expressed in factor intensive form to give $F(H_t, L_t)/L_t = f(h_t)$, where h_t (= H_t / L_t) is the per capita level of the harvest. The per capita production function has the standard properties: $f' > 0$ and $f'' < 0$. Furthermore, we assume $\lim_{h\to 0} f'(h_t) = \infty$ and $\lim_{h\to\infty} f'(h_t) = 0$.

The growth of the resource is $g(x_t)$, where x_t denotes the beginning of period t stock of the resource. $g(x_t)$ is assumed to be a strictly concave function, i.e. $g'' < 0$. Besides owning the stock the current old generation (generation t-1 in period t) will also get its growth, so that the stock they have available for trading is $x_t + g(x_t)$. Furthermore, following renewable resource economics, we assume that there are two values $x = 0$ and $x = \tilde{x}$ for which $g(0) = g(\tilde{x}) = 0$. Consequently, there is a unique value $\hat{x}$ at which

$g'(\hat{x}) = 0$. Hence, $\hat{x}$ denotes the stock providing the maximum sustained yield (MSY), while $\tilde{x}$ is the level at which growth is zero, i.e. the maximal stock that the natural environment can sustain. Moreover, it is reasonable to assume that there are no Inada conditions for the resource stock growth function. For instance, a logistic growth function ($g(x) = ax - (1/2)bx^2$) fulfills these assumptions.

The renewable resource in our model has two roles. It is both a store of value and an input in the production of the consumption good. The market for the resource operates in the following manner. At the beginning of the period the old agents own the stock, and also receive that period's growth of the stock. They sell the stock (growth included) to the firms, which then decide how much of that resource to harvest and use as an input in the production of the consumption good. The firm will sell the remaining stock of the resource to the young at the end of the period.

It is interesting to note that via growth function this "natural" production activity yields a profit for its owner. These profits could presumably be distributed in alternative ways. For instance, there could be a stock market where the ownership rights for the resource are exchanged. The young buy the shares for the resource, and when old, get the dividend and the proceeds from selling the shares next period.[8] This kind of arrangement leads to the same allocation, which we will have in our model.[9]

The transition equation for the resource is

(2) $$x_{t+1} = x_t - h_t + g(x_t),$$

where h_t denotes that part of the stock which has been harvested for use as an input in production. The initial stock and its growth, $g(x_t)$, can be conserved for the next period's stock or used for this period's harvest.

In addition to trading in the resource markets, the young can also participate in the financial markets by borrowing or lending, the amount of which is denoted by s_t. The periodic budget constraints are thus

(3) $$c_1^t + p_t x_{t+1} + s_t = w_t$$

(4) $$c_2^t = p_{t+1}\left[x_{t+1} + g(x_{t+1})\right] + R_{t+1} s_t$$

where p_t is the price of the resource and w_t is the wage rate in terms of period t's consumption, and $R_{t+1} = 1 + r_{t+1}$ is the interest factor. The young generation buys an amount

8 We are thinking here about the stock market arrangements proposed by Lucas (1978) and Brock (1982). Since they have infinitely lived agents, the treatment of the stock market in their papers cannot readily be applied to our overlapping generations model, where e.g. there is limited market participation. Brock (1990) presents an overlapping generations version of the asset pricing model of Lucas, where the asset pays a constant dividend each period. For a recent treatment of the stock market in an overlapping generations model with capital, see Magill and Quinzii (2001).

9 A sketch of the proof is available from the authors upon request.

x_{t+1} of the resource stock from the representative firm. The firm has harvested an amount h_t of the stock, and x_{t+1} has been left to grow. According to (4) the old generation consumes their savings including the interest, and the income they get from selling the resource next period to the firm, $p_{t+1}[x_{t+1} + g(x_{t+1})]$.

The periodic budget constraints (3) and (4) imply the lifetime budget constraint

$$(5) \qquad c_1^t + \frac{c_2^t}{R_{t+1}} = w_t + \frac{p_{t+1}[x_{t+1} + g(x_{t+1})] - R_{t+1} p_t x_{t+1}}{R_{t+1}}.$$

Maximizing (1) subject to (5) and the appropriate non-negativity constraints leads to the following first-order conditions for s_t and x_{t+1}

$$(6) \qquad u'(c_1^t) = R_{t+1} \beta u'(c_2^t)$$

$$(7) \qquad p_t u'(c_1^t) = p_{t+1}[1 + g'(x_{t+1})] \beta u'(c_2^t).$$

These conditions have straightforward interpretations. (6) is the Euler equation which says that the marginal rate of substitution between today's and tomorrow's consumption should be equal to the interest factor. According to (7) the marginal rate of substitution between consumptions in two periods should be equal to the resource price adjusted growth factor. (6) and (7) together imply the arbitrage condition for two assets

$$(8) \qquad R_{t+1} = [1 + g'(x_{t+1})] \frac{p_{t+1}}{p_t},$$

so that the interest factor is equal to the resource price adjusted growth factor.[10] Using (8) we can rewrite the lifetime budget constraint as

$$(9) \qquad c_1^t + \frac{c_2^t}{R_{t+1}} = w_t + \frac{p_{t+1}[g(x_{t+1}) - g'(x_{t+1}) x_{t+1}]}{R_{t+1}}.$$

The term in the square brackets is positive, since the growth function is strictly concave.

Next we define competitive equilibrium as follows.

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[10] Note that by choosing x the young can affect the marginal return of the resource, g'. This reflects the fact that a renewable resource, like fish stock, differs markedly from a conventional asset, whose return is independent of the amount invested.

Definition. A price system and a feasible allocation,

$\{p_t, R_t, w_t, c_1^t, c_2^{t-1}, h_t, x_t\}_{t=1}^{\infty}$ are competitive equilibrium, if

(i) given the price system, consumers and firms solve their decision problems and

(ii) markets clear for all t = 1,2,...,T,...

Market clearing conditions are

(10a) $c_1^t + c_2^{t-1} = f(h_t)$

(10b) $x_{t+1} + h_t = x_t + g(x_t)$

(10c) $s_t = 0$

(10d) $f'(h_t) = p_t$

(10e) $f(h_t) - h_t f'(h_t) = w_t$

(10a) is the resource constraint for all t, and (10b) is the transition equation for the renewable resource stock. The fact that there is only one type of a consumer per generation and no government debt forces the asset market clearing condition to be such that $s_t = 0$ for all t. Equations (10d) and (10e) in turn are the first-order conditions for profit maximization, and determine the evolution of factor prices, p_t and w_t.

The market clearing condition (10b) and the first-order condition (7) for the resource stock and harvesting together with the periodic budget constraints (3) and (4) and the equilibrium conditions (10d) and (10e) imply the following planar system that describes the dynamics of the model.

(11) $x_{t+1} = x_t - h_t + g(x_t)$

(12) $f'(h_t)u'[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}] =$

$$\beta f'(h_{t+1})u'[f'(h_{t+1})(x_{t+1} + g(x_{t+1}))]\ [1 + g'(x_{t+1})]$$

Equations (11) and (12) are the main objects of our study.[11] Before analyzing the qualitative properties of this system we characterize the steady state equilibria.

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[11] Instead of using Euler equation (12) we could have taken another route, as in Olson-Knapp (1997), for the dynamic analysis by concentrating on the evolution of total savings defined as $q(w_t, R_{t+1}, p_{t+1}, p_t) \equiv w_t - c_1^t(w_t, R_{t+1}, p_{t+1}, p_t)$. It is straightforward to show that $\partial q / \partial R_{t+1} < 0$, when the intertemporal elasticity of substitution is less than unity. See discussion below on the crucial importance of this elasticity in our analysis.

3. STEADY STATE EQUILIBRIA AND EFFICIENCY

In the steady state ($\Delta h_t = 0$ and $\Delta x_t = 0$) the following equations hold

(13) $$h = g(x)$$

(14) $$u'[f(h) - f'(h)h - f'(h)x] = \beta u'[f'(h)(x + g(x))]\,[1 + g'(x)].$$

Given the properties of the growth function, the curve defined by (13) is not monotone. Totally differentiating (14) we get

(15) $$\frac{dh}{dx} = \frac{\beta u'(c_2)g'' + \beta u''(c_2) f'(1+g')^2 + u''(c_1) f'}{-u''(c_1) f''(x+h) - \beta\, u''(c_2) f''(x+h)(1+g')} > 0.$$

The steady state in our model with general preferences and technology is not necessarily unique. When the growth rate, $g'(x)$, is positive, the Euler equation can cross the growth curve in many points. For steady state to be unique, it is necessary that the Euler equation cuts the growth curve from below. For growth rate $g'(x) \leq 0$ the steady state equilibrium is necessarily unique because of decreasing growth curve.

We next explore the existence and uniqueness of the steady state. The Euler equation (12) in the steady state reduces to

(16) $$\frac{u'[f(h) - f'(h)(h + x)]}{\beta u'[f'(h)(x + g(x))]} = 1 + g'(x).$$

First we consider the limit of the left-hand side, when the level of stock approaches the maximum steady state stock, $\tilde{x}$. Because $g(\tilde{x}) = 0$, the harvest is zero at $\tilde{x}$. Given the Inada conditions on the utility and production functions there must be a level of stock denoted by x^* such that

(17) $$\lim_{x \to x^*} \frac{u'[f(h) - f'(h)(h + x)]}{\beta u'[f'(h)(x + g(x))]} = \infty.$$

While the right-hand side is a decreasing function of x, since $g(x)$ is a strictly concave function, one cannot conclude that the left-hand side of (16) is a monotone function of x. The existence is guaranteed, if the left-hand side starts below the right-hand side at $x = 0$. It is easily seen that $\lim_{x \to 0} RHS(x) = \beta[1 + g'(0)]$. Without Inada conditions on the resource growth function, this term is a finite number. In particular, in the logistic case, there is a level of stock, say x', such that $1 + g'(x') = 0$.

To be able to say more about the existence, and in particular, the uniqueness of the steady state we consider a parametrized example. For the purposes of being able to tie the existence of the steady state to the values of economically interesting parameters we assume that the periodic utility function is of the form $u(c) = c^{1-\frac{1}{\rho}}/(1-(1/\rho))$, where ρ is the intertemporal elasticity of substitution. Using this utility function and the fact that in steady state $h = g(x)$ we rewrite equation (16) as

$$(18) \qquad \left(\frac{f'(h)h + f'(h)x}{f(h) - hf'(h) - xf'(h)}\right)^{\frac{1}{\rho}} = \beta\left[1 + g'(x)\right].$$

The part in parenthesis in the left-hand side can be written as follows

$$(19) \qquad \frac{f'(h)h + f'(h)x}{f(h) - hf'(h) - xf'(h)} = \frac{\frac{x}{h} + 1}{\frac{f}{f'h} - \frac{x}{h} - 1}.$$

If we have the Cobb-Douglas production function (i.e. $f(h) = h^{\alpha}$, with $0 < \alpha < 1$), then $f / f'h = 1/\alpha$, where the quantity $f / f'h$ is the elasticity of output with respect to harvest. Thus we can rewrite the LHS of (18) as

$$(20) \qquad LHS(x) = \left(\frac{\frac{x}{g(x)} + 1}{\frac{1}{\alpha} - \frac{x}{g(x)} - 1}\right)^{\frac{1}{\rho}}.$$

A straightforward differentiation yields

$$(21) \quad LHS'(x) = \frac{1}{\rho}\left(\frac{\frac{x}{g(x)} + 1}{\frac{1}{\alpha} - \frac{x}{g(x)} - 1}\right)^{\frac{1}{\rho}-1}\left(\frac{\frac{x}{g^2}(\frac{g}{x} - g')}{\frac{1}{\alpha} - \frac{x}{g} - 1} + \frac{(\frac{x}{g} + 1)\left[\frac{x}{g^2}(\frac{g}{x} - g')\right]}{(\frac{1}{\alpha} - \frac{x}{g} - 1)^2}\right).$$

Since the growth function is strictly concave, and provided that $\frac{1}{\alpha} - \frac{x}{g} - 1$ is positive, $LHS'(x) > 0$.[12]

Given the limit in (17) above it is sufficient for existence that $\lim_{x \to 0} LHS(x) < \lim_{x \to 0} RHS$. Now we can rewrite that condition as

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[12] Note that $c_1 = f(h) - f'(h)h - f'(h)x > 0$ in a steady state, then $f'h\left[\frac{f}{f'h} - \frac{x}{h} - 1\right] > 0$.

$$\text{(22)}\qquad LHS(0)=\left(\frac{\frac{1}{avg(0)}+1}{\frac{1}{\alpha}-\frac{1}{avg(0)}-1}\right)^{\frac{1}{\rho}}<\beta[1+g'(0)]=RHS(0),$$

where $avg(x)=g(x)/x$ evaluated at $x=0$ is finite as mentioned earlier. The economically interesting parameters are the elasticity of output with respect to harvest (α), discount factor (β), and the intertemporal elasticity of substitution (ρ). The higher the discount factor and the smaller the elasticity of output, the easier it is for the condition to hold. The effect of the intertemporal elasticity on condition (22) depends on the value of the term in parenthesis. If it is greater than unity, increasing the elasticity will decrease the LHS, and thus makes it easier to obtain the existence. If it is less than unity, increasing the elasticity will increase the LHS.[13] We depict the behavior of functions $LHS(x)$ and $RHS(x)$ in Figure 1.

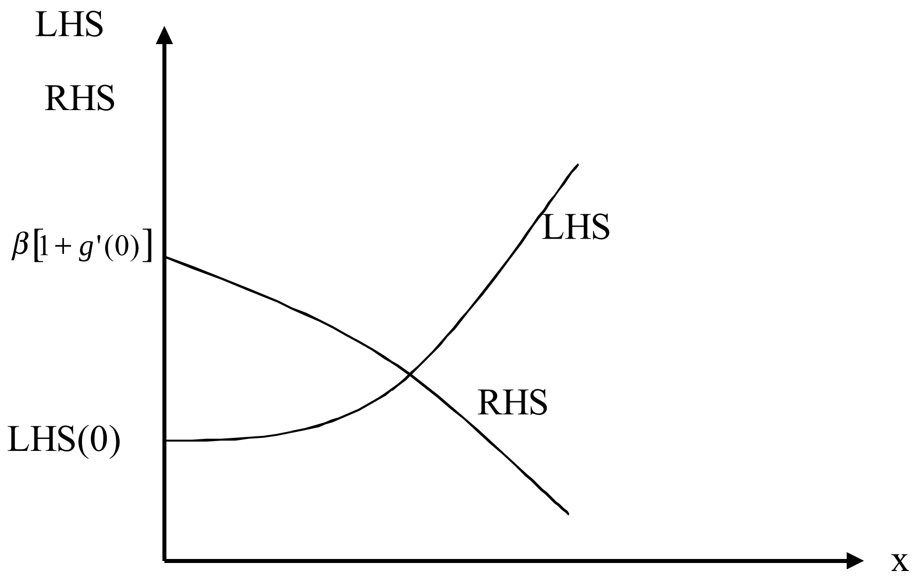

Figure 1. Existence and uniqueness of the steady state equilibrium.

We collect the previous discussion in

Proposition 1. *With iso-elastic utility function and Cobb-Douglas production* function, the existence of steady state depends on elasticity of output with respect to harvest, discount factor, and intertemporal elasticity of substitution. If steady

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[13] If we have the following logistic growth curve: $g(x)=x-(1/2)x^2$, then $avg(0)=1$. If $\alpha<1/4$, the term in brackets is less than unity. If $1/4<\alpha<1/2$, that term is bigger than unity.

state exists, it is unique with these spefications of the utility and production functions.

In the subsequent analysis we will concentrate on the nontrivial unique steady state.[14] We will describe the loci $\Delta x_t = 0$ and $\Delta h_t = 0$ in the hx-space. The slope of the locus, $h_t = g(x_t)$, evaluated at the steady state is

$$(23) \quad \left.\frac{dh_t}{dx_t}\right|_{\Delta x_t = 0} = g'(x),$$

and it can be positive, zero or negative. The slope of the locus (derived in Appendix 1) determined by equation (12), and evaluated at the steady state is

(24)

$$\left.\frac{dh_t}{dx_t}\right|_{\Delta h_t = 0} = \frac{u''(c_1)f'(1+g') + \beta u'(c_2)g''(1+g') + \beta u''(c_2)f'(1+g')^3}{u''(c_1)f' - u''(c_1)f''(x+g) + \beta u'(c_2)g'' - \beta u''(c_2)(1+g')\left[f''(x+g) - f'(1+g')\right]}$$

and is always positive given our assumptions on the utility function and the fact that $1+g'$ needs to be positive, because in the stationary equilibrium $1+g'$ equals the interest factor (c.f. arbitrage equation (8)).

The fact that we concentrate on the unique steady state implies that the following holds in steady state

$$(25) \quad \left.\frac{dh_t}{dx_t}\right|_{\Delta h_t = 0} > \left.\frac{dh_t}{dx_t}\right|_{\Delta x_t = 0}.$$

Thus the Euler equation cuts the growth curve from below.

Are the steady states efficient? It is a well-known fact that the competitive equilibria in overlapping generations models can be inefficient. Keeping in mind that $g'(x)$ is the rate of interest in the steady state and the population growth rate is zero in our model, we conclude that all those steady states for which $g'(x) \geq 0$ are efficient. This is the case where the real interest rate exceeds or equals population growth rate. Steady states in which $g'(x) < 0$ are inefficient, since consumption could be increased for every generation by harvesting some of the resource stock during any period. This case corresponds to the situation where the real interest rate is less than the population growth rate. This overaccumulation is inefficient.[15]

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[14] It can also be the case that the only point where the curves cross is the origin, especially, since we have not imposed Inada conditions on the growth function.

[15] Efficiency outside steady states is a more complicated problem. One can study the efficiency of nonstationary paths by modifying the criterion developed by Cass (1972) to the needs of the model at hand.

4. DYNAMICAL EQUILIBRIA: SADDLES

To study the qualitative properties of our model we start by considering paths for which $x_{t+1} \geq x_t$ and $h_{t+1} \geq h_t$. It follows from (11)

(26) $$x_{t+1} \geq x_t \Leftrightarrow x_t - h_t + g(x_t) \geq x_t \Leftrightarrow g(x_t) \geq h_t .$$

This means that x is increasing below the growth curve, and it is decreasing above the curve.

Considering paths for which $h_{t+1} \geq h_t$, requires more work. In Appendix 1 (equation A.3) we derive the following expression (evaluated at the steady state) for the derivative of the right-hand side of equation (12) above with respect to h_{t+1}

(27) $$\frac{\partial RHS}{\partial h_{t+1}} = (1+g')\beta f''u'\left(1-\frac{1}{\rho(c_2)}\right) \equiv A ,$$

where $\rho(c_2)$ $(= -[u'(c_2)/c_2u''(c_2)])$ is the reciprocal of the elasticity of the marginal utility of consumption. This is also known as the intertemporal elasticity of substitution, and it depends inversely on the curvature of the periodic utility function. Given the values of x_t and h_t, the right-hand side of equation (12) is an increasing (decreasing) function of h_{t+1}, if ρ is less (greater) than unity.

If $\rho > 1$ we get from (12)

(28) $$h_{t+1} \geq h_t \Leftrightarrow f'(h_t)u'[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}] \leq$$

$$\beta f'(h_t)u'[f'(h_t)(x_{t+1} + g(x_{t+1}))]\ [1 + g'(x_{t+1})]$$

Weak inequality, (28), is equivalent to the following statement

(29) $$\frac{u'[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}]}{\beta u'[f'(h_t)(x_{t+1} + g(x_{t+1}))]\ [1 + g'(x_{t+1})]} \leq 1$$

If $\rho < 1$, the inequalities in (28) and (29) are reversed. Therefore, the motion of h on both sides of the curve, where $h_{t+1} = h_t$, depends on the size of intertemporal elasticity of substitution. This fact points out to the possibility that dynamics of the system can drastically change when ρ passes through unity (logarithmic preferences). Intuitively, if the intertemporal elasticity of substitution is higher (lower) than one, then the substitution effect of the interest rate exceeds (falls short of) the income effect. Arbitrage condition (8) implies that more harvesting will increase (decrease) future consumption and gives the relationship in (28).

In order to study formally the stability properties of dynamical equilibrium, we first rewrite equation (11) as follows

(30) $\quad x_{t+1} = x_t - h_t + g(x_t) \equiv G(x_t, h_t)$

Substituting the RHS of (11) for x_{t+1} in (12) gives an implicit equation for h_{t+1},

(31) $\quad h_{t+1} = F(x_t, h_t)$

The planar system describing the dynamics of the renewable resource stock and harvesting consists now of equations (30) and (31). The Jacobian matrix of the partial derivatives of the system (11)-(12) can be written as

(32) $$J = \begin{bmatrix} G_x & G_h \\ F_x & F_h \end{bmatrix} = \begin{bmatrix} 1+g' & -1 \\ \dfrac{C}{A} & \dfrac{B}{A} \end{bmatrix},$$

where A has been derived above in equation (27) and B and C are the partial derivatives of equation (12) with respect to h_t and x_t respectively, and have been derived in Appendix 2. By defining $\hat{\rho} = \dfrac{\rho}{\rho - 1}$ the two ratios in the Jacobian matrix can then be expressed as

(33) $$\frac{C}{A} = \left\{ -\frac{f'^2 u''(c_1)}{\beta f'' u'(c_2)} - \frac{f'^2 u''(c_2)(1+g')^2}{f'' u'(c_2)} - \frac{f' g''}{f''} \right\} \hat{\rho}$$

(34) $$\frac{B}{A} = \left\{ 1 - \frac{f' u''(c_1)(x+h)}{u'(c_1)} + \frac{f'^2 u''(c_1)}{f'' u'(c_1)} + \frac{f'^2 (1+g') u''(c_2)}{f'' u'(c_2)} + \frac{f' g''}{f''(1+g')} \right\} \hat{\rho},$$

where we can see the importance of the magnitude of the intertemporal elasticity of substitution for the stability analysis. These elements of the Jacobian change signs whenever ρ passes through unity, since the bracketed term in C/A is negative and in B/A is positive.

The trace and determinant of the characteristic polynomial of our system can be calculated as

(35) $$D = (1+g')\hat{\rho} \left\{ 1 - \frac{f' u''(c_1)(x+h)}{u'(c_1)} \right\}$$

(36) $$T = (1+g') + \left\{ 1 - \frac{f' u''(c_1)(x+h)}{u'(c_1)} + \frac{f'^2 u''(c_1)}{f'' u'(c_1)} + \frac{f'^2 (1+g') u''(c_2)}{f'' u'(c_2)} + \frac{f' g''}{f''(1+g')} \right\} \hat{\rho}.$$

Armed with these calculations (see Appendix 2 for details) we get the following Proposition.

Proposition 2. *If the intertemporal elasticity of substitution is at least one half, but differs from unity, then all the steady state equilibria are saddle points*.

Proof. See Appendix 3.

According to Proposition 2, steady states are saddle points for a wide range of the values for intertemporal elasticity of substitution.[16] But when the intertemporal elasticity of substitution is unity, the periodic utility function is logarithmic, $u(c) = \ln c$, and the term $f'(h_{t+1})$ in equation (12) cancels out. It follows that the dynamical system reduces to a first-order nonlinear difference equation in the level of stock, x. Once the evolution of x is determined, the behavior of h can be obtained recursively. It can be shown that

Proposition 3. *When the utility function is logarithmic, the planar system reduces to a nonlinear first-order difference equation for the natural resource stock. If the steady state is unique, it is stable regardless of whether the equilibrium is efficient or not*.

Proof. See Koskela, Ollikainen and Puhakka (2002).

As Proposition 3 reveals, the assumption of logarithmic utility function in our model is special; saddle point equilibria vanish and stable equilibria emerge. Qualitatively though the properties of the equilibria with logarithmic preferences are very close to those of the saddle point (and thus determinate) equilibria. Since the initial condition for the resource stock is determined by history, the steady state and all the nonstationary equilibria tending towards it are determinate. An economic interpretation for Proposition 3 is the following. With logarithmic utility the substitution and income effects of the rate of return on consumption and saving will offset each other, so that changes in harvesting and resource stock via the arbitrage condition have no effect on Euler equation (12). Hence, the planar system reduces to natural resource stock dynamics and the system becomes recursive.

Empirical evidence on the size of the intertemporal elasticity of substitution does not, however, necessarily coincide with those parameter values presented in Propositions 2 and 3, but often points out to lower values.[17] It is therefore of interest to study also the characteristics of equilibria in the case when $\rho < 1/2$.

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16 For the saddle point one of the eigenvalues must be stable, i.e. it must be less than one in absolute value. Since the determinant is the product of eigenroots we can conclude that the stable root is negative if $D < 0$, and it is positive if $D > 0$.

17 See the discussion e.g. in Deaton (1991, pp. 63-75) and for a more recent survey by Attanasio (1999). Empirical evidence lies in conformity with the notion that the intertemporal elasticity of substitution is below one and might be less than one half.

5. DYNAMICAL EQUILIBRIA: INDETERMINACY AND FLIP BIFURCATIONS

In the above discussion we found that when $\rho > 1$, the determinant (D) and the trace (T) of the system are positive, and D-T+1 < 0. Steady states are thus saddles. These equilibria are in area C in Figure 2, in which we have reproduced the familiar graphical description of dynamical equilibria in a planar system (see e.g. Azariadis 1993, p. 66). Stable equilibria lie in area B, and the other saddle poin equilibria are in area A. Thus complex roots are not possible in our model.

We are interested in seeing whether our model displays bifurcations. When $\rho < 1$, the determinant of the system becomes negative, and D-T+1 positive. This means that the saddle-node bifurcations, which require among other things that D-T+1 = 0, are not possible. We already proved that steady state equilibria are saddles for $1 > \rho \geq 1/2$. Since D+T+1 cannot be unambiguously signed for $\rho < 1/2$, it is possible to have stable equilibria and flip bifurcations (see areas A and B in Figure 2).

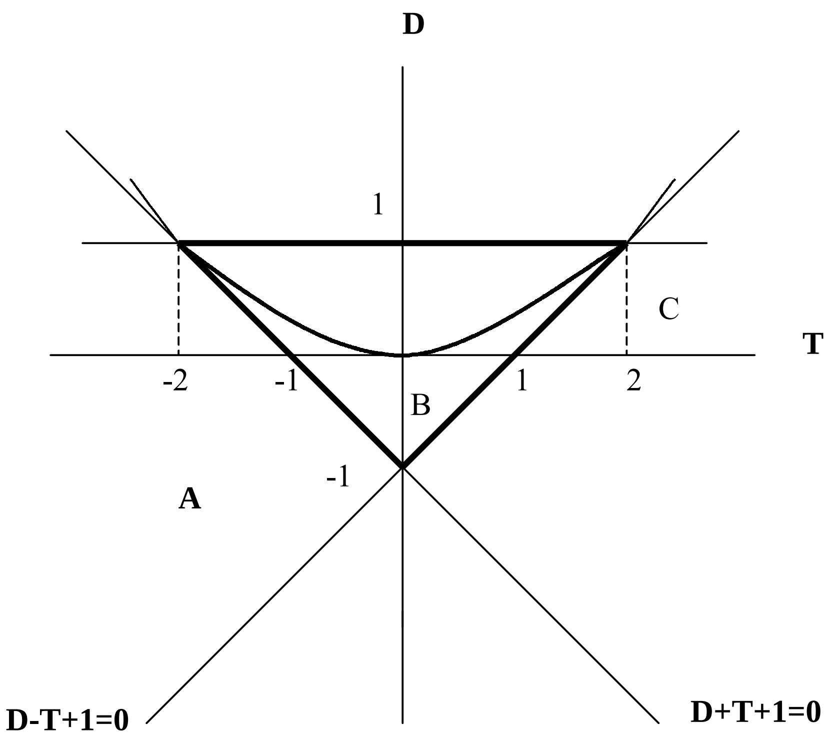

Figure 2. Characteristics of stability in a planar system

Assuming $\rho < 1/2$ we have D < 0. We established in Proposition 1 that, when $1 < \rho \geq 1/2$ D-T+1 > 0. To get stability, we need to have D+T+1 > 0 as well. Because we have rigorously shown the existence of saddles when D < 0 (area A in Figure 2), we can also show the existence of flip bifurcations, if we can show the stability of equilibria. To proceed we rewrite D+T+1 as follows

(37) $\quad D = (1+g')\hat{\rho}\{M+1\}$

(38) $\quad T = (1+g') + \hat{\rho}\{M+N+1\}$,

where $M = -\dfrac{f'u''(c_1)(x+h)}{u'(c_1)} > 0$

$$N = \left\{ \frac{f'^2\, u''(c_1)}{f''u'(c_1)} + \frac{f'^2\,(1+g')u''(c_2)}{f''u'(c_2)} + \frac{f'g''}{f''(1+g')} \right\} > 0 .$$

Using this notation we can express D+T+1 after some manipulation as

(39) $\quad$ D+T+1 = $(2+g')\hat{\rho}M + \hat{\rho}N + (2+g')(1+\hat{\rho})$.

This shows that at least in principle D+T+1 can be zero or positive, if the last term, the only positive term in the expression, dominates. Note that when D <0, D-T+1 > 0 and D+T+1 = 0 we have a flip bifurcation (see the line between areas A and B in Figure 2).

Providing analytical results on bifurcations is easier for a one-dimensional model. Since our planar system is quite complex, mainly due to the nonlinear gross return from investing in the natural resource, we conjecture that reducing the dimension of our planar system to one by finding the center manifold (applying the Center Manifold Theorem) for our model may not be a straightforward task.[18] In this section we consider a parametric example with logistic growth function and standard explicit functional forms for utility and production:

$$u(c) = \frac{c^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}};\ \rho \neq 1,\ f(h) = h^{\alpha};\ g(x) = ax - \frac{1}{2}bx^2$$

where ρ is the intertemporal elasticity of substitution. In the steady state, $h = ax - (1/2)bx^2$. Using this expression for h, the Euler equation and budget constraints, we end up with the following expression (see Appendix 4) for the stock of the renewable resource in a steady state equilibrium

(40) $$\frac{1}{1+(1+a-bx)^{\rho}\beta^{\rho}} + \frac{\alpha}{a-\frac{1}{2}bx} = 1-\alpha .$$

A straightforward but tedious calculation yields the following expression for D+T+1

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[18] On the Center Manifold Theorem, see Guckenheimer and Holmes (1986), p. 127. For an example of finding the center manifold in a different two-dimensional model, see Reichlin (1992).

(41) $$D+T+1=\left(\frac{1}{\rho-1}\right)\frac{(2+a-bx)\alpha(1+a-\frac{1}{2}bx)}{\left[(1-\alpha)(a-\frac{1}{2}bx)-\alpha\right]}+(2+a-bx)\left(\frac{1-2\rho}{1-\rho}\right)$$

$$+\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{\rho-1}\right)\left[\alpha\left[1+(1+a-bx)^{\rho}\beta^{\rho}\right]+\frac{\alpha(1+a-bx)^{1-\rho}\left[1+(1+a-bx)^{\rho}\beta^{\rho}\right]}{\beta^{\rho}}+\frac{\rho b(ax-\frac{1}{2}bx^{2})}{1+a-bx}\right]$$

In the sequel we undertake a numerical analysis for the parametric example of our model. We assume the following values for parameters of the growth function and the discount factor: $a=b=1$ and $\beta=1/2$.[19] The values for growth parameters mean that $\hat{x}=1$ and $\tilde{x}=2$, and furthermore that the condition $1+g'(x)\geq 0$ holds for all $0\leq x\leq 2$. Hence we use the assumptions of the growth function, which in the one-dimensional case eliminate more complex dynamics, and are in this sense "well-behaved".[20] Economically interesting parameters are the output elasticity of resource (α), which determines the price elasticity of resource demand, and the intertemporal elasticity of substitution (ρ). For this reason our focus will be on finding out for what values of these parameters we will get stability and flip bifurcations.

Solving α from equation (40) and plugging that value into (41) we find out for what combinations of x and ρ D+T+1 is greater or less than zero or exactly zero. Solving α from (40) we get

(42) $$\alpha=\frac{2a-bx}{2+2a-bx}-\frac{2a-bx}{2+2a-bx}\left(\frac{1}{1+(1+a-bx)^{\rho}\beta^{\rho}}\right).$$

Plugging this relationship into (41) gives the following expression

(43) $$D+T+1=\left(\frac{1}{\rho-1}\right)(2+a-bx)(1+a-bx)^{\rho}\beta^{\rho}+(2+a-bx)\left(\frac{1-2\rho}{1-\rho}\right)$$

$$+\left(\frac{1}{\rho-1}\right)\left[\frac{\rho bx(2a-bx)(2+2a-bx)\left[1+(1+a-bx)^{\rho}\beta^{\rho}\right]}{2(1+a-bx)\left[2a-bx+2\left[1+(1+a-bx)^{\rho}\beta^{\rho}\right]\right]}\right]$$

$$+\left(\frac{1}{\rho-1}\right)\left[\frac{(2a-bx)(1+a-bx)^{\rho}\beta^{\rho}(1+(1+a-bx)^{\rho}\beta^{\rho})}{2a-bx+2\left[1+(1+a-bx)^{\rho}\beta^{\rho}\right]}\right]+$$

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[19] If we want to interpret literally the length of the period in our overlapping generations economy to be around 25 years, then the annual discount factor 0.975 (or the rate of time preference about 2.6 percent) means that the discount factor for 25 years should be around ½.

[20] If the value of the parameter a varies, and in particular, is above three, then a period doubling bifurcation and chaos start to emerge even in this one dimensional logistic equation. Since this has been thoroughly studied in the existing literature on nonlinear dynamics, we use a parameter value for a, which in itself does not produce bifurcations or chaotic behavior, see e.g. Holmgren (1996).

$$+\left(\frac{1}{\rho-1}\right)\left[\frac{(2a-bx)(1+a-bx)(1+(1+a-bx)^{\rho}\beta^{\rho})}{2a-bx+2\left[1+(1+a-bx)^{\rho}\beta^{\rho}\right]}\right].$$

To get a more precise idea where to look for stable equilibria, note that the only positive term in this expression is the second term. Combining this term and the first term we get after rearranging

$$(44)\qquad \left(\frac{2+a-bx}{1-\rho}\right)\left[(1-2\rho)-(1+a-bx)^{\rho}\beta^{\rho}\right].$$

Consider first the efficient allocations, which lie on the left-hand side of the maximum sustained yield, i.e. $0 \le x \le a/b$. It is quite straightforward to see that the term in the brackets of (44) is negative. This means that all the steady states are saddles. Therefore, we should look for possible stable equilibria from the right-hand side of the MSY, where equilibrium is inefficient.

For given a, b and ρ, the stationary equilibrium condition (40) indicates that there is an inverse relationship between α and x. Because we will now concentrate on such allocations for which $x > a/b$, the value of α must be relatively small for equation (40) to hold.

Our approach will be the following. We will first graph the surface defined by equation (43) in the (D+T+1) $x\rho$ - space. Then we set D+T+1 = 0, and graph those values of x and ρ for which D+T+1 = 0 holds.

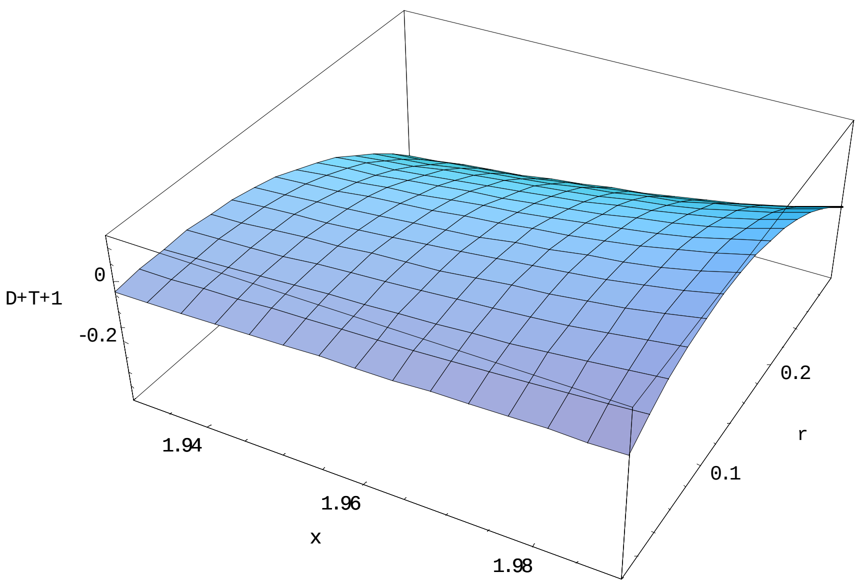

Figure 3. D+T+1.

Figure 3 is the three-dimensional graph of equation (43) (when α has been substituted in for the expression of D+T+1). It points out to the fact that D+T+1 will be positive only for extremely high (i.e. values which are close to $\tilde{x}$ (= 2)) levels of the renewable resource stock.

In Figure 4 we have projected those values of the resource stock x and the elasticity of intertemporal substitution ρ for which D+T+1 is exactly zero, i.e., for which we have flip bifurcations. Values of x and ρ, which lie above the curve will yield stable equilibria, and for the values of x and ρ below the curve we have saddle point equilibria.

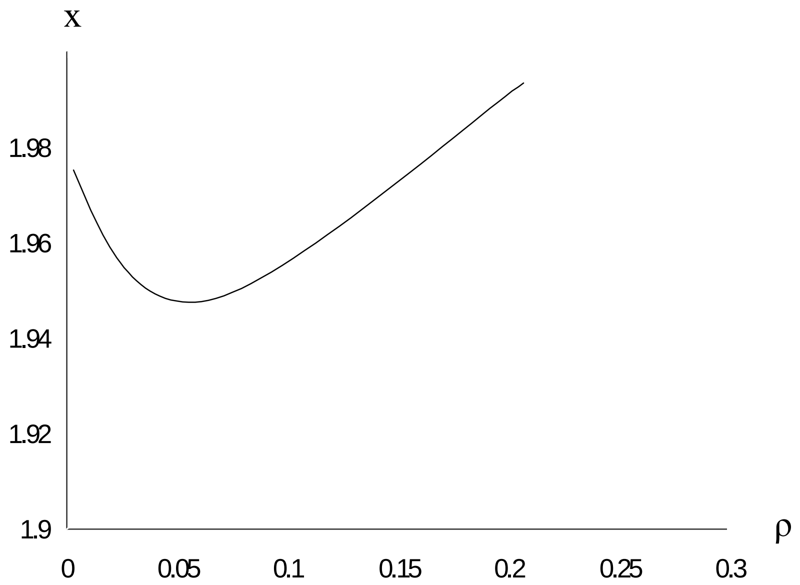

Figure 4. A characterization of indeterminacy and flip bifurcations

In Figure 5 we have depicted α, x and ρ in the same diagram, i.e. we have graphed equation (42). This figure indicates that to get stable equilibria and flip bifurcations the value of α needs to be quite small. Hence there is a set of parameter values of α and ρ, for which our parametrized economy exhibits stable equilibria, i.e., indeterminacy and flip bifurcations.[21] This also means that there can be endogenous cycles in our model, since the characteristic roots are of different sign.

There are two types of flip bifurcations. In a supercritical flip bifurcation a stable two-cycle emerges on the side of the bifurcation value, where the steady state is a saddle. In a subcritical bifurcation, an unstable two-cycle emerges on the side of the bifurcation value, where the steady state is stable.

[21] E.g. if $\alpha = 0.01$ and $\rho = 0.03$ we get the level of the steady state stock to be 1.95664 and the level of harvesting 0.04242. We also get $D+T+1 = 0.00119886$. If instead we let $\alpha = 0.011$, we get the equilibrium stock to be 1.95228, the level of harvesting 0.04658, and $D+T+1 = -0.00373852$.

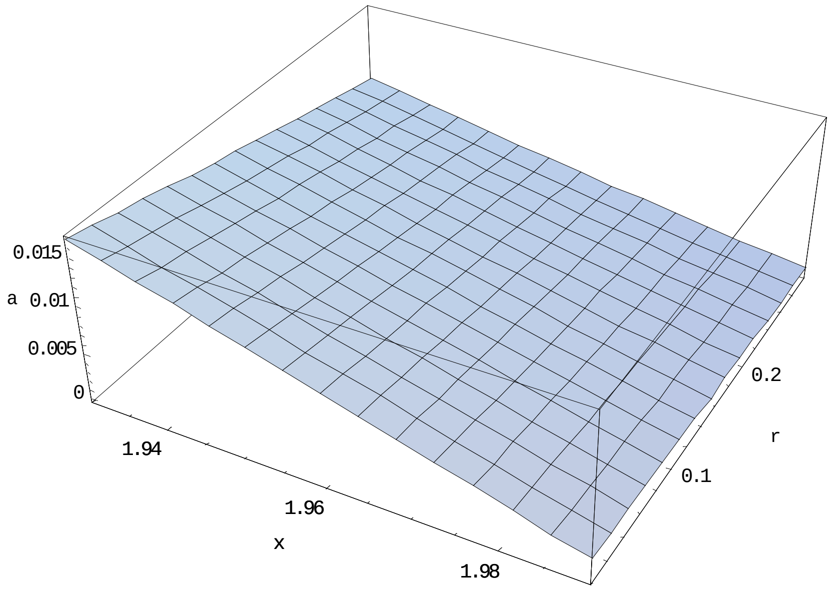

Figure 5. Equation (42).

To investigate the type of flip bifurcation, i.e. on which side of the flip bifurcation a two-cycle exists in our model, we resort to numerical simulations, and ask is it possible to find four numbers $\{x_1, h_1; x_2, h_2\}$, which solve the following transition and Euler equations?

(45) $$x_2 = x_1 - h_1 + x_1 - \frac{1}{2}x_1^2 \quad \text{and} \quad \frac{h_1^{\alpha-1}}{\left[(1-\alpha)h_1^{\alpha} - \alpha\, h_1^{\alpha-1} x_2\right]^{\frac{1}{\rho}}} = \frac{\beta\, h_2^{\alpha-1}(2-x_2)}{\left[\alpha\, h_2^{\alpha-1}(x_1+h_2)\right]^{\frac{1}{\rho}}}$$

(46) $$x_1 = x_2 - h_2 + x_2 - \frac{1}{2}x_2^2 \quad \text{and} \quad \frac{h_2^{\alpha-1}}{\left[(1-\alpha)h_2^{\alpha} - \alpha\, h_2^{\alpha-1} x_1\right]^{\frac{1}{\rho}}} = \frac{\beta\, h_1^{\alpha-1}(2-x_1)}{\left[\alpha\, h_1^{\alpha-1}(x_2+h_1)\right]^{\frac{1}{\rho}}}$$

If we find a four-tuple that fulfills equations (45) and (46), then a two-cycle exists. We fixed $\alpha = 0.004$, and chose the values of the intertemporal elasticity of substitution from both sides of the Flip bifurcation curve in Figure 4. In Figure 6 we have chosen to depict the emergence of the two-cycle for the resource stock x (the vertical axis).

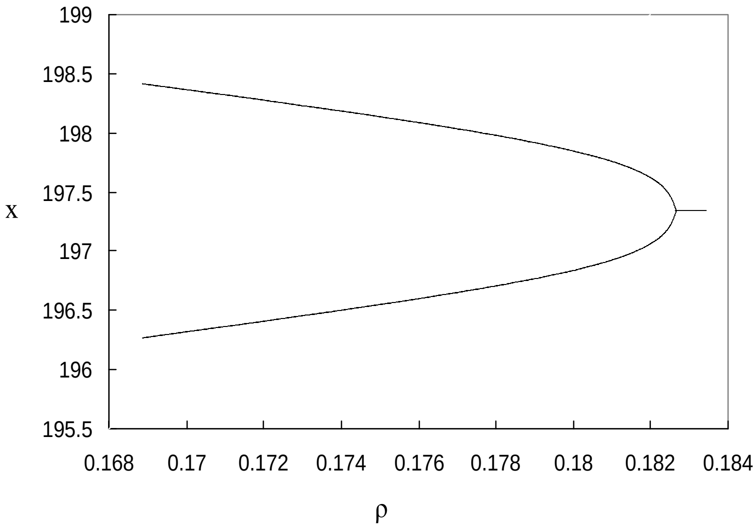

Figure 6. Subcritical flip bifurcation.

The flip bifurcation occurs for values of ρ (the horizontal axis) between 0.1825 and 0.1826. If ρ = 0.1826, we have a saddle, and if it is 0.1825 we have a stable equilibrium. As the Figure indicates the period doubling occurs on the stable side, which means that we have a subcritical flip bifurcation. We are now in a position to summarize our findings in

Proposition 4. *If the intertemporal elasticity of substitution is less than half, and the logistic resource growth function is "well-behaved", there exists stable equilibria (indeterminacy) and a subcritical bifurcation.*

Flip bifurcations and complex dynamics emerge in our model with standard overlapping generations structure due to the mixture of low elasticity of intertemporal substitution in consumption and a well-behaved logistic growth. The parameter values for the intertemporal elasticity of substitution for which we get stability and flip bifurcations are empirically quite plausible. The parameter values of the share of the resource in total output, α, are quite small. This seems to be quite plausible when one considers, for instance, exploitation of fish stocks in typical fishing nations such as Spain and Norway in Europe; Canada, Peru and Chile in North and South America, or Japan in Asia.

6. CONCLUSIONS

The stability properties of an overlapping generations model with capital accumulation, like periodic equilibria and indeterminacy of equilibria, have been subject to a fairly large amount of research since the mid 1980s. These issues have not, however, been studied carefully in models with renewable resource use, like fisheries. This is somewhat surprising in the light of the fact that exploited fish populations seldom exhibit steady state yields, and more likely provide cyclical, irregular or spasmodic yields. In this paper we have examined the dynamic properties of an overlapping generations economy with standard assumptions about the utility and production functions, but augmented with a renewable resource, which is a factor of production and serves as a store of value.

We showed that the nature of the steady state equilibrium depends on the value of intertemporal elasticity of substitution in consumption. In particular, if the intertemporal elasticity of substitution is at least one half, but different from unity, then steady states are saddle points. Interestingly, for smaller values of the intertemporal elasticity of substitution, which are plausible on the basis of empirical evidence from consumption behavior, we use a parametric example to demonstrate the existence of a subcritical flip bifurcation for the case of inefficient equilibrium. This means that a repelling two-cycle emerges on the side of flip bifurcation where the steady state is stable. When the intertemporal elasticity of substitution is equal to one, the dynamical system reduces to a first-order nonlinear difference equation in the resource stock and is stable.

We have demonstrated a possibility for different types of equilibrium dynamics in a standard overlapping generations model with a renewable resource. Interestingly, these various properties of dynamical equilibria lie in conformity with empirics from the exploited fish populations, which exhibit not only the conventionally analyzed steady yields, but also cycles and irregularities.

Appendix 1. The RHS of equation [12] as a function of h_{t+1}, and the derivation of equation [24]

- The right-hand side of equation (12) as a function of h_{t+1}.

The RHS of (12) is

A.1 $$RHS(h_{t+1}) = \beta f'(h_{t+1})u'\left[f'(h_{t+1})(x_{t+1} + g(x_{t+1}))\right]\left[1 + g'(x_{t+1})\right]$$

Differentiating this with respect to h_{t+1} we get (dropping the arguments when convenient)

A.2 $$RHS'(h_{t+1}) = (1+g')\beta f''u' + (1+g')\beta f'u''\left[f''(x+g(x))\right]$$

$$= (1+g')\beta f''\left[u' + f'(x+g(x))u''\right]$$

Keeping in mind that $c_2 = f'(x+g(x))$ we have

A.3 $$RHS'(h_{t+1}) = (1+g')\beta f''u'\left(1 - \frac{1}{\rho(c_2)}\right)$$

where $\rho(c_2) = \dfrac{-u'(c_2)}{cu''(c_2)}$ is the elasticity of intertemporal substitution. From A.3 it is now easy to see that $RHS'(h_{t+1}) > 0\,(< 0)$ when $\rho(c_2) < 1\,(> 1)$.

- The derivation of equation (24)

We first rewrite equation (12), and take into account the fact that we consider paths, where $h_{t+1} = h_t$ for all t but x may vary.

A.4 $$u'\left[f(h_t) - f'(h_t)h_t - f'(h_t)x_{t+1}\right] = \beta u'\left[f'(h_t)(x_{t+1} + g(x_{t+1}))\right](1 + g'(x_{t+1}))$$

Totally differentiating A.4 and taking into account equation (10) we get

A.5 $$\{\ u''(c_1^t)\left[-f''(x_t + g(x_t)) + f'\right] + \beta u'(c_2^t)g''(x_{t+1}) -$$

$$\beta u''(c_2^t)\left[f''(x_{t+1} + g(x_{t+1})) + f'(1 + g'(x_{t+1})\right](1 + g'(x_{t+1}))\ \}dh_t$$

$$=$$

$$\{\ u''(c_1^t)f'(1 + g'(x_t) + \beta u'(c_2^t)g''(x_{t+1})(1 + g'(x_t)) +$$

$$\beta u''(c_2^t)\left[f'\left[1 + g'(x_t) + g'(x_{t+1})(1 + g'(x_t))\right]\right](1 + g'(x_{t+1}))\ \}dx_t.$$

Rearranging and evaluating A.5 at the stationary point, $h_{t+1} = h_t$ and $x_{t+1} = x_t$, yields equation (24) in the text.

Appendix 2. Development of the Jacobian Matrix of the Partial Derivatives

For the purposes of stability analysis we develop the Jacobian matrix, its determinant and trace.

A.6 $x_{t+1} = G(x_t, h_t)$

A.7 $x_{t+1} = F(x_t, h_t)$

The stability of the steady state depends on the eigenvalues of the Jacobian matrix of the partial derivatives

$$J = \begin{bmatrix} G_x & G_h \\ F_x & F_h \end{bmatrix}.$$

Calculating the partial derivatives of the Jacobian matrix we first obtain

$$G_x(x_t, h_t) = 1 + g'(x_t),\ G_h(x_t, h_t) = -1.$$

To get the partials of $h_{t+1} = F(x_t, h_t)$ we first do the implicit differentiation in the following manner

A.8 $Adh_{t+1} = Bdh_t + Cdx_t$,

where A, B and C are appropriate partial derivatives to be presented in a moment. Calculating these we take into account $x_{t+1} = x_t - h_t + g(x_t)$. Given the definitions of A, B and C we will then have

$$F_x(x_t, h_t) = \frac{C}{A},\ F_h(x_t, h_t) = \frac{B}{A}.$$

As for A (as evaluated at the steady state) we get from A.3

A.9 $A = (1 + g')\beta f''u'(c_2)\dfrac{\rho - 1}{\rho}$.

For the future developments we define $\hat{\rho} = \dfrac{\rho}{\rho - 1}$. Clearly, $A > (<)0$, as $\rho < (>)1$. Totally differentiating (12) with respect to h_t (again taking into account the transition equation) we obtain

A.10 $B = f''(h_t)u'(c_1^t) + f'(h_t)u''(c_1^t)\left[- f''(h_t)(x_t + g(x_t)) + f'(h_t)\right] +$

$$\beta\left[f'(h_{t+1})\right]^2 u''(c_2^t)\left[1 + g'(x_{t+1})\right]^2 + \beta f'(h_{t+1})u'(c_2^t)g''(x_{t+1}) < 0,$$

and totally differentiating (12) with respect to x_t (again taking into account the transition equation) we have

A.11 $C = -[f'(h_t)]^2 u''(c_1^t)[1+g'(x_t)] - \beta f'(h_{t+1})u'(c_2^t)g''(x_{t+1})[1+g'(x_t)] -$

$$\beta[f'(h_{t+1})]^2 u''(c_2^t)[1+g'(x_t)]\ [1+g'(x_{t+1})]^2 > 0.$$

Next we evaluate A, B and C at the steady state. By taking into account the Euler condition at the steady state $u'(c_1) = (1+g')\beta u'(c_2)$, we get

A.12i $\dfrac{C}{A} = \left\{ -\dfrac{f'^2 u''(c_1)}{\beta f'' u'(c_2)} - \dfrac{f'^2 u''(c_2)(1+g')^2}{f'' u'(c_2)} - \dfrac{f' g''}{f''} \right\} \hat{\rho}$

A.12ii $\dfrac{B}{A} = \left\{ 1 - \dfrac{f' u''(c_1)(x+h)}{u'(c_1)} + \dfrac{f'^2 u''(c_1)}{f'' u'(c_1)} + \dfrac{f'^2 (1+g') u''(c_2)}{f'' u'(c_2)} + \dfrac{f' g''}{f''(1+g')} \right\} \hat{\rho}$.

Clearly, $C/A > (<)0$ when $\rho < 1(>1)$, and $B/A > (<)0$ when $\rho > 1(<1)$.

We can now rewrite the Jacobian as follows

A.13 $J = \begin{bmatrix} 1+g' & -1 \\ \dfrac{C}{A} & \dfrac{B}{A} \end{bmatrix}$.

The determinant (D) and the trace (T) of the Jacobian matrix, J, are D = $(1+g')\dfrac{B}{A} + \dfrac{C}{A}$ and T = $1+g' + \dfrac{B}{A}$ respectively. Using equations A.9, A.10 and A.11 we have the following expressions

A.14 $D = (1+g')\hat{\rho}\left\{ 1 - \dfrac{f' u''(c_1)(x+h)}{u'(c_1)} \right\}$

A.15 $T = (1+g') + \left\{ 1 - \dfrac{f' u''(c_1)(x+h)}{u'(c_1)} + \dfrac{f'^2 u''(c_1)}{f'' u'(c_1)} + \dfrac{f'^2 (1+g') u''(c_2)}{f'' u'(c_2)} + \dfrac{f' g''}{f''(1+g')} \right\} \hat{\rho}$.

Appendix 3. Proof of Saddle Point Stability

We analyze the stability of system (30) and (31) on the basis of (11) and (12).

The characteristic polynomial associated with the system (30) and (31) expressed in terms of D and T is

A.16 $p(\lambda) = \lambda^2 - T\lambda + D = 0$

It is known from the stability theory of difference equations (see e.g. Azariadis, 1993, pp. 63-67) that for a saddle point to exist the roots of $p(\lambda) = 0$ need to be on both sides of (minus and plus) unity. Thus for a saddle we need that D-T+1 < 0 and D+T+1 > 0 or D-T+1 > 0 and D+T+1 < 0.

When $\hat{\rho}$ is positive ($\rho > 1$) both the determinant and the trace in A.14 and A.15, respectively, are positive, which also means that D+T+1 > 0 holds. Making inferences about the sign of D-T+1 requires work. A straightforward calculation yields

A.17 D-T+1=

$$g'(\hat{\rho}-1) + \hat{\rho}\left\{-\frac{f'u''(c_1)(x+h)g'}{u'(c_1)} - \frac{f'^2 u''(c_1)}{f''u'(c_1)} - \frac{f'^2(1+g')u''(c_2)}{f''u'(c_2)} - \frac{f'g''}{f''(1+g')}\right\},$$

so that A.17 cannot be signed yet for $\hat{\rho} > 0$. To get the sign of D-T+1 we use the assumption that our steady state is unique. This is assured by comparing slopes of the curves, where $h_{t+1} = h_t$ and $x_{t+1} = x_t$. We develop the condition

A.18 $$\left.\frac{dh_t}{dx_t}\right|_{\Delta h_t = 0} > \left.\frac{dh_t}{dx_t}\right|_{\Delta x_t = 0},$$

as

A.19 $$\frac{u''(c_1)f'(1+g') + \beta u'(c_2)g''(1+g') + \beta u''(c_2)f'(1+g')^3}{u''(c_1)f' - u''(c_1)f''(x+h) + \beta u'(c_2)g'' - \beta u''(c_2)(1+g')[f''(x+h) - f'(1+g')]} > g'.$$

Multiplying both sides of A.19 by the denominator (negative sign) on the left-hand side we get

A.20 $$u''(c_1)f'(1+g') + \beta u'(c_2)g''(1+g') + \beta u''(c_2)f'(1+g')^3 <$$

$$u''(c_1)f'g' - u''(c_1)(x+h)f''g' + \beta u'(c_2)g''g' - \beta u''(c_2)(1+g')f''(x+h)g'$$

$$+\beta u''(c_2)(1+g')^2 f'g'.$$

and collecting terms A.20 can be re-expressed as

A.21 $u''(c_1)f'+\beta u'(c_2)g''+\beta u''(c_2)f'(1+g')^2+u''(c_1)(x+h)f''g'$

$+\beta u''(c_2)(1+g')f''(x+h)g'<0$.

Dividing by ($f''\beta u'(c_2)<0$), using Euler condition and the fact that $c_2 = f'(x+h)$ yields

A.22
$$\frac{u''(c_1)f'(1+g')}{f''u'(c_1)}+\frac{g''}{f''}+\frac{u''(c_2)f'(1+g')^2}{f''u'(c_2)}+\frac{u''(c_1)(x+h)g'(1+g')}{u'(c_1)}-\frac{1}{\rho}\frac{(1+g')g'}{f'}>0.$$

Now we multiply both sides by $f'/(1+g')$ (>0) to get

A.23 $$\frac{u''(c_1)f'^2}{f''u'(c_1)}+\frac{f'g''}{f''(1+g')}+\frac{u''(c_2)f'^2(1+g')}{f''u'(c_2)}+\frac{u''(c_1)(x+h)g'f'}{u'(c_1)}-\frac{1}{\rho}g'>0.$$

Rearranging and taking into account the definition of $\hat{\rho}$ yields

A.24 $$(\frac{\hat{\rho}-1}{\hat{\rho}})g'+\left\{-\frac{f'u''(c_1)(x+h)g'}{u'(c_1)}-\frac{f'^2u''(c_1)}{f''u'(c_1)}-\frac{f'^2(1+g')u''(c_2)}{f''u'(c_2)}-\frac{f'g''}{f''(1+g')}\right\}<0.$$

If $\hat{\rho}>0$ (i.e. $\rho>1$) we get by multiplying with $\hat{\rho}$

A.25 $$g'(\hat{\rho}-1)+\hat{\rho}\left\{-\frac{f'u''(c_1)(x+h)g'}{u'(c_1)}-\frac{f'^2u''(c_1)}{f''u'(c_1)}-\frac{f'^2(1+g')u''(c_2)}{f''u'(c_2)}-\frac{f'g''}{f''(1+g')}\right\}<0.$$

Note that this is exactly D-T+1, which means that we have a saddle when $\rho>1$.

If $\hat{\rho}<0$ (i.e. $\rho<1$) we get by multiplying with $\hat{\rho}$

A.26

$$g'(\hat{\rho}-1)+\hat{\rho}\left\{-\frac{f'u''(c_1)(x+h)g'}{u'(c_1)}-\frac{f'^2u''(c_1)}{f''u'(c_1)}-\frac{f'^2(1+g')u''(c_2)}{f''u'(c_2)}-\frac{f'g''}{f''(1+g')}\right\}>0$$

which means that D-T+1 is positive. To get a saddle in this case, we need to have D+T+1 < 0. To explore this possibility when $\hat{\rho}<0$ we rewrite D and T as follows

A.27i $D=(1+g')\hat{\rho}\{M+1\}$

A.27ii $T=(1+g')+\hat{\rho}\{M+N+1\}$,

where

$$M = -\frac{f'u''(c_1)(x+h)}{u'(c_1)} > 0$$

$$N = \left\{ \frac{f'^2\, u''(c_1)}{f''u'(c_1)} + \frac{f'^2\,(1+g')u''(c_2)}{f''u'(c_2)} + \frac{f'g''}{f''(1+g')} \right\} > 0 .$$

Using this shorthand notation D+T+1 can be expressed after some manipulation

A.28 D+T+1 = $(2+g')\hat{\rho}M + \hat{\rho}N + (2+g')(1+\hat{\rho})$.

The first two terms in (A.28) are negative, when $\hat{\rho} < 0$. The third term is also negative when $1+\hat{\rho} < 0$. This happens when $\rho > 1/2$. So we have a saddle in this case, too. This completes the proof of Proposition 1. Q.E.D.

Appendix 4. Derivation of equation (40)

Given the assumed functional forms, the Euler equation can be written

A.29 $c_2 = [(1+g')\beta]^{\rho} c_1$.

Plugging this into the equilibrium condition, $c_1 + c_2 = f(h)$ and using the budget constraint $c_2 = f'(h)(x+g(x))$ gives

$$c_1 = \frac{[x(a-(1/2)bx)]^{\alpha}}{1+(1+a-bx)^{\rho}\beta^{\rho}} \text{ and } c_2 = \frac{\alpha[x(a-(1/2)bx)]^{\alpha}[1+a-(1/2)bx]}{(a-(1/2)bx)}.$$

If we plug these expressions for consumption back into the equilibrium condition we get equation (40) in the text.

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