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ON THE PROBABILISTIC RELATIONSHIP BETWEEN THE PUBLIC GOOD INDEX AND THE NORMALIZED BANZHAF INDEX**

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ABSTRACT: In this paper we develop a probabilistic interpretation for the normalized Banzhaf index and the public good index. We then derive a way to decompose the normalized Banzhaf index into two parts. The first of them relates the Banzhaf index with the public good index and the other one on special type of luck. Similarities and differences of the two indices are then discussed.

Keywords: power indices, voting games

SUMMARY

In this paper, we give a probabilistic interpretation for normalized Banzhaf index and (normalized) public good index. Both indices can be interpreted as conditional probabilities and their difference as a special luck concept. Both indices can also be characterized with public good conditions. If that is done, the paper argues that NBI assumes that powerful players gain from free-riders whereas PGI presumes that they do not gain. The essential difference between the two indices is hence based on the treatment of surplus players, i.e. on players who are not able to swing within a crucial coalition. These members of a crucial coalition create an externality which makes some swing players to be able to swing more often than in the case that surplus players are not considered.

The paper shows that, despite the fact that BI is based on private good interpretation of policies, NBI is necessarily not. In fact PGI and NBI can be put under a similar probability model, which can be expressed either in terms of public or private good interpretation. The basis for the common model is non-normalized Banzhaf index. The paper demonstrates that the normalization of BI is not so innocent a transformation as it is often thought. Moreover, we argue that the normalization adds new information to the model of BI indicating that we are talking about a different model. The most crucial consequence is that it is not so straightforward how BI should be normalized to obtain reasonable estimates for relative power or should it be normalized at all.

Recently, it has been shown that there are two alternative ways to approach public good index: to define an outcome as a public good and presume that surplus players do not exert power or to build a simple bargaining model where bargaining power is due to the membership in minimal winning coalitions. In this paper, we have shown that there exists a third story, which, in fact, has elements of both previous interpretations. Our characterization is based on probabilistic voting and, in some sense, rational expectations as excess coalitions are not formed. This, in fact, shows that public good index is more of ex post nature than the traditional power indices.

The paper also illustrates that a probabilistic interpretation of standardized indices leads to somewhat artificial assumptions. Especially this holds for NBI as it is difficult to imagine a rationale for the assumption that only crucial coalitions will form. In contrast, the assumption that only minimal winning coalitions will form, sounds more reasonable at least in ex post terms. If there is perfect information on players' preferences after a proposal is made one might argue that surplus players do not have incentives to join a minimal winning coalition as her pay-off would be zero. In other words, the policy outcome of a minimal winning coalition should be the same as the policy outcome of a minimal winning coalition with additional surplus players.

1 Introduction

In this paper, we build a probability model for two normalized power indices: the *normalized Banzhaf index* (NBI)[1] and the *public good index* (PGI).[2] The former is usually interpreted simply as a rescaled version of *non-normalized Banzhaf index* (BI),[3] which has its basis in a well-established probability model.[4] Moreover, BI presumes that players who exert power divide the spoils of an outcome equally. It is thus based on private good interpretation of policies.

In this paper, we take a different view. The paper shows that, despite the fact that BI is based on private good interpretation of policies, NBI is necessarily not. In fact PGI and NBI can be put under a similar probability model, which can be expressed either in terms of public or private good interpretation. The basis for the common model is non-normalized Banzhaf index. The paper demonstrates that the normalization of BI is not so innocent a transformation as it is often thought. Moreover, we argue that the normalization adds new information to the model of BI indicating that we are talking about a different model. The most crucial consequence is that it is not so straightforward how BI should be normalized to obtain reasonable estimates for relative power or should it be normalized at all.

One focus of this paper is to make a distinction between voting and policy outcomes and show that there is potentially a substantial difference between the power distributions according to the chosen approach. To illustrate the difference we derive a decomposition to show that voting influence can be presented as a weighted sum of policy power and luck. This decomposition gives us more information concerning the nature of the ways to influence in decision making.

In Barry (1980) an actor's success in a voting game is decomposed into power and luck. In his terminology, success measures how well the voting outcome corresponds with a player's views, i.e. her most preferred outcome and the decomposition shows that it is due to power and luck. In Barry's decomposition power does not mean traditional power indices but an actor's

[1] See Banzhaf (1965).

[2] For its derivation, see Holler and Packel (1983).

[3] See Penrose (1946) and Banzhaf (1965).

[4] For a more detailed discussion, see Owen (1972), Straffin (1977, 1988), Leech (1990) or Felsenthal and Machover (1998).

influence on policy outcomes.

More recently the power index correspondence to Barry's conceptualization was defined and analyzed by Holler and Packel (1983). Their PGI differs from standard power indices in the sense that in the latter an actor exerts power when she is either pivotal to a minimal winning coalition (Shapley-Shubik index, SSI, Shapley and Shubik 1954) or critical to a minimal winning coalition, like in BI. In this context, minimal winning coalitions are defined as coalitions where there exists at least one player who can turn the coalition from winning to losing.[5] A crucial difference between SSI and BI is that former sums up to unity over players and latter does not. In empirical studies BI is often normalized. However, if we take a probabilistic view on power indices then this normalization is not without problems (see Straffin 1977) since it destroys the probabilistic voting model that is behind BI. In this paper, we elaborate more on this issue and demonstrate that NBI and PGI have substantial similarities as they are actually restricted versions of the probability model that is behind BI. NBI and PGI require more information on coalition formation than BI.

In their recent book, Felsenthal and Machover (1998) distinguish between, as they call, I-power and P-power. The former is based on policy-seeking viewpoint, hence being proportional to voter's influence on an outcome of a vote. The latter is based on office-seeking, hence being proportional to voter's control of the ultimate outcome or, in other words, expected share in the fixed purse. Felsenthal and Machover argue that BI and NBI represent I-power, the latter being only a rescaling of the former. In this paper we argue that the normalization of BI is not just a rescaling but rather a way to add information to the model. Basically, the same holds for PGI.

In the next section we first derive a probabilistic interpretation to NBI and PGI. Then under probabilistic interpretation we find that the two indices belong to the same family of indices. Their most essential difference lies in the assumption of which coalitions will form.[6]

[5] More recently the public good index has been studied by Holler and Li (1995), Holler (1998). From a bargaining perspective the public good index has been derived by Brams and Fisburn (1995).

[6] For a recent survey on power indices' similarities and differences, see Laruelle (1999).

2 Probabilistic models for normalized indices

2.1 Basic framework

To distinguish between voting and policy outcomes we first define an explicit relationship between the two. Let N be a set of n players in a voting game (N, v) where v is an indicator function $v : \mathcal{P}(N) \rightarrow \{0, 1\}$ and let us denote by S an arbitrary coalition, $S \subset N$. There are 2^n possible coalitions. Let us denote the class of coalitions by 2^N. For each vote we may write *the result vector* R of ones and zeros where one as the i^{th} element corresponds with player i's yes-vote and zero with a no-vote. Let $\mathcal{W}$ denote the class of winning coalitions (majorities) formally defined as follows: $\{S \in \mathcal{W}\} \Leftrightarrow \{S \in 2^N,\ \mu(S) \geq q\}$ where $\mu(S)$ denotes the number of votes in coalition S and q denotes the required majority rule. In terms of the result vector, we can define a majority result as follows $\{R \in \mathcal{W}'\} \Leftrightarrow \{R \in \{0,1\}^n,\ w^T R \geq q\}$ where $\mathcal{W}'$ is the set of majority result vectors and w^T is the transpose of a vector of voting weights and q the share of votes needed for a majority. Let us also denote the cardinalities of sets by small letters, like S has s players, and cardinalities of classes of sets by $|\mathcal{S}|$ where $\mathcal{S}$ is a class of sets.

Definition 1 *A coalitional form of a voting game can be defined as*

$$v(S) = \begin{cases} 1, & \textit{if } S \in \mathcal{W} \\ 0, & \textit{if } S \notin \mathcal{W} \end{cases}$$

Traditional power indices assume that a player exerts power whenever her vote makes a difference to a voting outcome. This happens when a voter is able to swing a losing coalition into a winner or when she is able to change the status of a majority result vector into a minority result vector. Let us denote the class of losing coalitions by $\mathcal{L}$. The power index is the probability that a voter is able to make the difference.

2.2 Probabilistic interpretation of the NBI

Let us next interpret the normalized Banzhaf index with a simple probability model. We take the probability model for non-normalized Banzhaf index as the basis of our model. Then, contrast to Felsenthal and Machover (1998), we argue that the normalization of BI is not an innocent way to rescale the index but rather a way to add extra information to the model. Therefore, NBI is not based on maximal entropy like it is the case with BI.[7]

Definition 2 *Suppose $S \subset N$ a randomly chosen coalition and let $\mathcal{W}$ and $\mathcal{L}$ denote the classes of winning and losing coalitions, respectively. Then the class of crucial coalitions (CWC), $\mathcal{C}$, is formed by winning coalitions with the following property*

$$\{S \in \mathcal{C}\} \Leftrightarrow \{S \in \mathcal{W},\ \exists i \in S:\ S - \{i\} \in \mathcal{L}\}.^{8}$$

Definition 3 *The class of crucial coalitions w.r.t. i, $\mathcal{C}_i$, is formed by coalitions with the following property*

$$\{S \in \mathcal{C}_i\} \Leftrightarrow \{S \in \mathcal{W},\ S - \{i\} \in \mathcal{L}\}.$$

Note that $\mathcal{C}_i \subset \mathcal{C}$ and $\bigcup\limits_{i \in N} \mathcal{C}_i = \mathcal{C}$ and $\sum\limits_{i \in N} |\mathcal{C}_i| \geq |\mathcal{C}|$.

The class $\mathcal{C}_i$ defines the coalitions where i is critical for a majority result since if i changes his vote from 1 (yes) to 0 (no) the majority result vector turns to a minority result vector.. Note that player i exerts voting power in coalitions in $\mathcal{C}_i$. If $\mathcal{C}_i$ is empty player i is a dummy player.

The basic idea behind the Banzhaf index is to evaluate voters' a priori prospects of wielding influence in decision making behind a veil of ignorance. Therefore it sounds reasonable to assume that a priori she is assumed to be indifferent between a yes- and no-vote. Hence for player i, $P(R_i = 1) = P(R_i = 0) = \frac{1}{2}$.

Write

Definition 4 *If $P(R_i = 1) = P(R_i = 0) = \frac{1}{2}$ for all $i \in N$ we say that players are indifferent.*

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[7]For a more detailed discussion, see Felsenthal and Machover 1998, 35-38.

[8]Note that Deegan & Packel (1979) use the term minimal winning coalitions for the elements of $\mathcal{C}$. Here we follow Bolger's (1979) conceptualization with this respect. Usually minimal winning coalitions are defined as coalitions such that there are no winning coalitions among their proper sub-sets. The class of crucial coalitions consists of coalitions that have at least one losing coalition as a proper sub-set. Felsenthal and Machover (1998) use the term vulnerable coalitions.

The motivation behind this simple choice is twofold. First, it serves as a neutral base for our comparison and, second, when power is evaluated in an abstract sense, as here, there is basically no reason to give more weight for certain outcomes.

Remark 1 *Note that indifference implies that all coalitions are equally likely to occur because* $P(R_i = 1) = \frac{1}{2} = 1 - P(R_i = 1)$ *for all* i.

By symmetry it is without loss of generality to concentrate on coalitions where i is a member (votes 'yes'). As there are 2^{n-1} coalitions where i belongs to the maximal number of crucial coalitions w.r.t. i is 2^{n-1}.[9] Now, the non-normalized Banzhaf index can be written as a share of crucial coalitions w.r.t. i of all coalitions where i belongs to. Hence

$$\beta_i' = \frac{|\mathcal{C}_i|}{2^{n-1}}. \tag{1}$$

Multiplying and dividing (1) by $2^{|\mathcal{W}|-1}$ we obtain

$$\beta_i' = \frac{|\mathcal{C}_i|}{2^{|\mathcal{W}|-1}} \frac{2^{|\mathcal{W}|}}{2^n}.$$

The first term of this decomposition gives the probability of being a swing player given that i belongs to a winning coalition and the second term gives the probability of forming a winning coalition. The latter term is a decreasing function of majority rule, the sum of non-normalized Banzhaf indices over the set of players can be interpreted in terms of status quo bias. Banzhaf index is usually normalized as follows

$$\beta_i = \frac{\beta_i'}{\sum_{i \in N} \beta_i'} = \frac{|\mathcal{C}_i|}{\sum_{i=1}^{n} |\mathcal{C}_i|}. \tag{2}$$

Note that the standardization destroys the above-described probabilistic interpretation of the non-normalized index which takes into account the difficulties in reaching a majority (see Straffin 1988). The normalized index answers simply the question of, what is voter i's share of all possible critical swings. Giving an equal weight to each crucial coalition and each swing

[9] Note that this is also the maximal number of losing coalitions that i can swing to become crucial w.r.t. i.

gives us the classical probability in (2). Hence NBI can be interpreted as a conditional probability of being a swing player given that each crucial set is equally likely to occur, one of them is formed and each swing is equally like. The list of conditions reveals that probabilistic interpretation of NBI is not without problems. By assuming independence we may, however, argue that any crucial coalition is equally likely as any other. As the same argument holds for any coalition $C - \{i\}, C \in \mathcal{C}\ \forall i \in N$ each swing is equally likely given that a crucial coalition is formed.

Proposition 1 *The normalized Banzhaf index gives a probability of being crucial if*

- voters are indifferent

and the following private good conditions hold

(a) a crucial coalition is formed

(b) each swing within a crucial coalition has the same likelihood

(c) the actual swing player obtains the whole (fixed) value of a winning coalition.

or the following public good conditions hold

(a') only crucial coalions are considered for measuring power

(b') the value of a crucial coalition is valid for all members of the coalition

Note that the non-normalized Banzhaf index gives estimates for players' probabilities of swinging a winning coalition into losing if they are unconditionally indifferent. This implies that all coalitions have the same probability of occurrence.

As there are coalitions with no crucial players, hence no swings, the non-normalized Banzhaf index does not sum up to unity. The denominator of equation (1) gives the number of coalitions that can be swinged by player i. Since there are coalition that cannot be swinged it is straightforward that the index does not sum up to unity. In probabilistic terms, the essential properties of the normalization are given by conditions (a)-(c). These conditions give more information on how the gains of winning coalitions are allocated among their members. We may also think that the conditions form a bargaining model and bargaining takes place when the actual proposal for voting is made. Therefore, normalized indices are not such as a priori measure as non-normalized ones. Condition (a) states that only crucial coalitions will form. It may sound odd to restrict coalition formation so significantly but it is worth noting that by symmetry we can deal with crucial coalitions that are either for or against a proposal.[10] Condition (b) implies that each proper losing sub-coalition of a crucial coalition has the same probability of occurrence, which is a straightforward implication of the indifference assumption. Condition (c), basically, states that swing players in a crucial coalition have equal outside options. In general, the gains are divided proportionate to the number of swing positions that the players have. The swing player who actually makes a critical swing obtains the whole value of a winning coalition but, a priori, all swing players in one crucial coalition are in symmetric position.

Alternatively we may take a public good view on NBI. This is shown by conditions (a') and (b'). They indicate that only crucial coalitions are considered for measuring power. In practise, this means that the outcome of the game must be different if different crucial coalitions are formed. Condition (b') simply states that the value of a coalition is available for all its members, hence the gains (or losses) that are the consequence of an outcome will not be shared by the members of a coalition. Consider an outcome of a vote. The public good conditions of the NBI show that all members of a crucial coalition get the full gain of the outcome since it was supposed to be a public good. In crucial coalitions there are, however, players who do not contribute to the outcome since they are not needed for the decision. These players gain only by being members of that particular crucial coalition. We may therefore argue that if we take the public good view on the NBI we also assume that free-riders create an externality, which makes crucial players power exaggregated.

From the private good point of view conditions (a)-(c) indicate that voters are not indifferent any more in general sense but only if a crucial coalition is formed. This is due to the extra information about the coalition formation. But how reasonable are these conditions? The most straightforward criticism

[10]Note that voting games in coalitional form do not model agenda setting. A proposal is chosen randomly.

stems from the fact that there are *surplus players* in crucial coalitions. A surplus player is a player who is not able to swing, hence from the private good point of view a player who will get no gain under any circumstances from being a member of a crucial coalition. Why should one think that he would become member of such coalition in the first place. Let us illustrate this with a simple example.

Consider the following five-person voting game $[51; 30, 30, 20, 10, 10]$ where 51 stands for the majority requirement and the vector (30,30,20,10,10) stands for a vector of individual votes. Following the definition of $\mathcal{C}$ both coalitions $\{30, 20, 10, 10\}$ and $\{30, 20, 10\}$ are crucial with respect to the 30-vote member. The policy outcome of these two coalitions should be, however, the same since the 30-vote member does not need both 10-vote members to accomplish a decision as $(1, 0, 1, 1, 1)$ and $(1, 0, 1, 1, 0)$ are both majority results. Hence one 10-vote member is a surplus player and the definition of $\mathcal{C}$ overestimates 30-vote members' influence since the same policy outcome has been taken twice into consideration. Hence, if we interpret power as an ability to affect the pursued policies we have to distinguish between the crucial coalitions with and without surplus players.

In regard with common policies of a certain organization, 'assigning the value of an outcome to the swing-voter' approach may not be proper. For example, in the common policies of the European Union the gains of policy outcomes are not confined to those who voted on the winning side nor are the losses confined to those who were on the losing side (Barry 1980). The policy of the Union is the same for each member and whether it is good or bad for a country is according to her situation. The NBI confines the gains of a voting outcome to those who are critical to the outcome in the above-mentioned sense. This is due to the construction of the NBI by considering the class $\mathcal{C}_i$ for measuring power.

2.3 The Probabilistic Interpretation of the PGI

If we consider the value of a coalition to be collective good, any member of the voting body whose preferences correspond with the winning outcome can be considered as a member of a specific coalition in $\mathcal{W}$. However, only those who are decisive for the policy can exert power (i.e. influence on the policy). The rest of the coalition members can be interpreted as free-riders who should not be considered when one evaluates how much influence players' wield in policy outcomes. Since in the context of the NBI the surplus players can make other voters critical for a result it follows that it is more reasonable to restrict class of coalitions further when evaluating voters' influence (Holler and Packel 1983). This leads to the exclusion of surplus players.

Definition 5 *Let us denote by $\mathcal{M} \subset 2^N$ the class of coalitions defined as follows:*

$$\{S \in \mathcal{M}\} \Leftrightarrow \{S \in \mathcal{W}, S - \{i\} \in \mathcal{L} \ \forall i \in S\}.$$

The class $\mathcal{M}$ is referred to as minimal winning coalitions (MWC). Particularly, let us denote by $\mathcal{M}_i$ the minimal winning coalitions where i is a member.

Measuring power by using MWCs eliminates surplus players from the computation since it is presumed that any superset of a minimal winning coalition pursues the same policy. Applying the same method of conditionalizing BI as above we end up to the following power index, which is better known as PGI

$$\theta_i = \frac{|\mathcal{M}_i|}{\sum_{i=1}^{n} |\mathcal{M}_i|}. \tag{3}$$

Now, PGI can be interpreted as a classical probability for i's swing given three conditions as follows

Proposition 2 *Public good index gives probability of exerting power if*

- voters are indifferent

and the following *private good conditions* hold

(d) a minimal winning coalition is formed

(e) each swing within a minimal winning coalition has the same likelihood

(f) the actual swing player obtains the whole value of a winning coalition.

or the following *public good conditions* hold

(d') only minimal winning coalions are considered for measuring power (e') the value of a minimal winning coalition is valid for all members of the coalition

PGI can be interpreted as player i's conditional probability of being a swing player, hence wielding influence, given conditions (d)-(f). Alternatively we can characterize PGI by picking only the minimal winning coalitions and then power is determined by the players' memberships in minimal winning coalitions. Public good conditions are given by (d') and (f').

2.4 A Comparison of NBI and PGI

It is easy to see that the essential difference between (2) and (3) stems from the results that are taken into account. Probabilistic interpretation of NBI assumes that a crucial coalition will form whereas probabilistic interpretation of PGI assumes that minimal winning coalitions will form. To justify the latter we may think that surplus players do not have incentives to join minimal winning coalitions since their pay-off would be zero.

To give a heuristic model for NBI and PGI we may think that players in a voting body are indifferent and try to figure out what kind of coalitions will form when an issue of vote arises. If players are not able to make any further predictions of their influence on an outcome they make their evaluation on the basis of BI. They thus approximate the their probabilities of being crucial by their probabilities of being a swing player but as a group their estimates are inconsistent. There is simply not enough information to form consistent estimates. To make them consistent there is a temptation to normalize BI. But, as it is demonstrated above, this normalization changes the probability model of voting. If one assumes that a crucial coalition will form and conditions (b) and (c) hold normalized Banzhaf index is an appropriate proxy for players' influence. Conditions (a)-(c) form, however, an additional bargaining model for a voting situation. In particular, as formation of crucial coalitions is very difficult to justify in the light of bargaining theory, we argue that NBI is not a reasonable measure of power. The bargaining conditions that make the trick of normalization are not credible.

The main difference between PGI and NBI stems from the prediction of which coalitions will form. With this respect the former seems more reasonable. Minimal winning coalitions do not contain surplus players. To characterize the difference more in depth we may write

Proposition 3 *The relation between NBI and PGI can expressed by the following decomposition*

$$\beta_i = (1 - \pi) \cdot \theta_i + \pi \cdot \varepsilon_i \tag{4}$$

where $\varepsilon = \frac{|\overline{C}_i|}{\sum_{j \in N} |\overline{C}_i|}$ *and* $\pi = \frac{\sum_{i=1}^{n} |\overline{C}_i|}{\sum_{i=1}^{n} |C_i|}$ *and* $\overline{C}_i$ *denotes a subclass of crucial coalitions* C_i *with the following property:*

$$\left\{S \in \overline{C}_i\right\} \Leftrightarrow \{S \in C_i \ \wedge \exists j \in S \ s.t. \ S - \{j\} \in \mathcal{W}_i.\}$$

i.e. S is crucial but not minimal winning.

Proof. (see appendix).

The interpretation of π is straightforward. It expresses the game-specific share of crucial coalitions in which there are surplus players. It is easy to see that $\pi < 1$ as there exists at least one minimal winning coalition in any majority game. If $\pi = 0$ there are no surplus players at all. If this holds NBI equals PGI, which in fact indicates that we are dealing either with a unanimity game or a symmetric game in the sense that all players have the same amount of power.[11] Term π can be interpreted as an overall measure (share) of potential luck. It gives the share of coalitions that are crucial but not minimal winning. Term ε_i varies from individual to individual. It can be interpreted as a relative share of luck that a player has, hence as an normalized index of luck. It measures players' shares of swings that are due to the externality of surplus players.[12]

Consider the following example [3;3,1,1]. We have four crucial coalitions A, AB, AC and ABC and one minimal winning coalition A. Here normalized Banzhaf index gives (1,0,0) since there are four swings from ABC to BC, from

[11]The simplest form of these games gives one vote for each voter. Note, however, that the game [51;49,49,2] belongs to this class of games as well. Unanimity games with any allocation of votes also have this property.

[12]Note that luck here has a different meaning from Barry's (1980) conceptualization. In his terminology, luck stems from the cases where the outcome of a vote corresponds with one's views although she is not able to influence on the outcome. Surplus players may have luck. In this paper, surplus players' luck creates an externality and the players who are able to gain from it are lucky. This externality should not, however, make a difference to a the policy outcome.

AB to A, from AC to A and from A to $\emptyset$. We have $|\mathcal{M}_A| = 1$, $|\mathcal{M}_B| = 0$, $|\mathcal{M}_C| = 0$; $|\mathcal{C}_A| = 4$, $|\mathcal{C}_B| = 0$, $|\mathcal{C}_C| = 0$ and $\left|\overline{\mathcal{C}}_A\right| = 3$, $\left|\overline{\mathcal{C}}_B\right| = 0$, $\left|\overline{\mathcal{C}}_C\right| = 0$. From these we get $\theta = (1,0,0)$, $\varepsilon = (1,0,0)$ and $\pi = \frac{3}{4}$. In this example, players B and C are surplus players in all crucial coalitions they belong to.

Consider next the example of a five-person simple majority voting game $[51; 30, 30, 20, 10, 10]$. In Table 1, the first row shows the numbers of votes and the rows 2-11 give the crucial coalitions of. Minimal winning coalitions are marked with $\mathcal{M}$ on the first column. Three last rows of the table give the numbers of crucial and minimal winning coalitions where player i is able to swing. Let us denote the player set by $ABCDE$.

	30	30	20	10	10
$\mathcal{M}$	X	X			
	X	X	X		
	X	X		X	
	X	X			X
	X	X		X	X
$\mathcal{M}$	X		X	X	
$\mathcal{M}$	X		X		X
$\mathcal{M}$		X	X	X	
$\mathcal{M}$		X	X		X
	X		X	X	X
		X	X	X	X
$\|\mathcal{M}_i\|$	3	3	4	2	2
$\|\mathcal{C}_i\|$	8	8	7	6	6
$\|\overline{\mathcal{C}}_i\|$	5	5	3	4	4

Table 1. An example of the decomposition of NBI

Using (4) and the respective numbers in Table 1 we obtain

$$
\begin{aligned}
\beta_A &= \left(1 - \frac{21}{35}\right)\left(\frac{3}{14}\right) + \left(\frac{21}{35}\right)\left(\frac{5}{21}\right) = \frac{8}{35} = \beta_B \\
\beta_C &= \left(1 - \frac{21}{35}\right)\left(\frac{4}{14}\right) + \left(\frac{21}{35}\right)\left(\frac{3}{21}\right) = \frac{7}{35} \\
\beta_D &= \left(1 - \frac{21}{35}\right)\left(\frac{2}{14}\right) + \left(\frac{21}{35}\right)\left(\frac{4}{21}\right) = \frac{6}{35} = \beta_E.
\end{aligned}
$$

The example also shows that PGI is not necessarily monotonic in voting weight.[13]

In the earlier literature, there are two different stories behind PGI (see Holler 1998 for details). First, there is the original public good interpretation, which assumes that whatever winning coalition will form only those who belong to a decisive subset of a winning coalition exert power since any superset of a decisive set is formed by luck. The value of a winning coalition is a public good, hence computation is based on membership (see Holler 1982, Holler and Packel 1983). Second, there is a formally identical index, which is introduced by Brams and Fishburn (1995) and which is also based on minimal winning coalitions. Players' relative bargaining power depends on how many memberships they have in minimal winning coalitions. The main difference between the PGI and member MWC-index is that the former does not assume that only minimal winning coalitions will form whereas member MWC-index does. With this respect this paper gives the third story for PGI. Our story is closer to the original PGI since it counts swings. The similarity between our story and Brams and Fishburn (1995) stems from the assumption that a minimal winning coalition will form. The same difference holds for the difference of the usual story behind NBI and our probabilistic model.

3 Conclusions

We gave a probabilistic interpretation for normalized Banzhaf index and (normalized) public good index. Both indices can be interpreted as conditional probabilities and their difference as a special luck concept. Both indices can also be characterized with public good conditions. If that is done, the paper argues that NBI assumes that powerful players gain from free-riders whereas PGI presumes that they do not gain. The essential difference between the two indices is hence based on the treatment of surplus players, i.e. on players who are not able to swing within a crucial coalition. These members of a crucial coalition create an externality which makes some swing players to be able to swing more often than in the case that surplus players are not considered.

[13] For a discussion, see Holler (1997).

Recently, it has been shown that there are two alternative ways to approach public good index: to define an outcome as a public good and presume that surplus players do not exert power or to build a simple bargaining model where bargaining power is due to the membership in minimal winning coalitions. In this paper, we have shown that there exists a third story, which, in fact, has elements of both previous interpretations. Our characterization is based on probabilistic voting and, in some sense, rational expectations as excess coalitions are not formed. This, in fact, shows that public good index is more of ex post nature than the traditional power indices.

The paper also illustrates that a probabilistic interpretation of standardized indices leads to somewhat artificial assumptions. Especially this holds for NBI as it is difficult to imagine a rationale for the assumption that only crucial coalitions will form. In contrast, the assumption that only minimal winning coalitions will form, sounds more reasonable at least in ex post terms. If there is perfect information on players' preferences after a proposal is made one might argue that surplus players do not have incentives to join a minimal winning coalition as her pay-off would be zero. In other words, the policy outcome of a minimal winning coalition should be the same as the policy outcome of a minimal winning coalition with additional surplus players.

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Appendix

Let us define a new class of winning coalitions, $\overline{\mathcal{C}}_i$, describing the difference of crucial and minimal winning coalitions

$$\{S \in \overline{\mathcal{C}}_i\} \Leftrightarrow \{S \in \mathcal{C}_i,\ \exists j \in S:\ S - \{j\} \in \mathcal{W}_i\}. \tag{5}$$

It is easy to see that

$$\mathcal{M}_i \cap \overline{\mathcal{C}}_i = \emptyset \tag{6}$$

and

$$\mathcal{M}_i \cup \overline{\mathcal{C}}_i = \mathcal{C}_i. \tag{7}$$

Hence

$$\sum_{i \in N} |\mathcal{C}_i| = \sum_{i \in N} |\mathcal{M}_i| + \sum_{i \in N} |\overline{\mathcal{C}}_i|. \tag{8}$$

Using the definitions in section 2.4 the difference $\beta_i - \theta_i$ can be written as follows

$$\begin{aligned}
\frac{|\mathcal{C}_i|}{\sum_{i=1}^n |\mathcal{C}_i|} - \frac{|\mathcal{M}_i|}{\sum_{i=1}^n |\mathcal{M}_i|} &= \frac{|\mathcal{C}_i| \sum_{i=1}^n |\mathcal{M}_i| - |\mathcal{M}_i| \sum_{i=1}^n |\mathcal{C}_i|}{\sum_{i=1}^n |\mathcal{M}_i| \sum_{i=1}^n |\mathcal{C}_i|} \\
&= \frac{(|\mathcal{M}_i| + |\overline{\mathcal{C}}_i|) \sum_{i=1}^n |\mathcal{M}_i| - |\mathcal{M}_i| \sum_{i=1}^n |\mathcal{M}_i| + \sum_{i=1}^n |\overline{\mathcal{C}}_i|}{\sum_{i=1}^n |\mathcal{C}_i| \sum_{i=1}^n |\mathcal{M}_i|} \\
&= \frac{|\overline{\mathcal{C}}_i| \sum_{i=1}^n |\mathcal{M}_i| - |\mathcal{M}_i| \sum_{i=1}^n |\overline{\mathcal{C}}_i|}{\sum_{i=1}^n |\mathcal{M}_i| \sum_{i=1}^n |\mathcal{C}_i|} \\
&= \frac{\frac{|\overline{\mathcal{C}}_i|}{\sum_{i=1}^n |\overline{\mathcal{C}}_i|} \sum_{i=1}^n |\mathcal{M}_i| - |\mathcal{M}_i|}{\frac{\sum_{i=1}^n |\mathcal{M}_i| \sum_{i=1}^n |\mathcal{C}_i|}{\sum_{i=1}^n |\overline{\mathcal{C}}_i|}} \\
&= \pi \cdot \varepsilon_i - \pi \cdot \theta_i
\end{aligned}$$

where $\varepsilon = \frac{|\overline{\mathcal{C}}_i|}{\sum_{j \in N} |\overline{\mathcal{C}}_i|}$ and $\pi = \frac{\sum_{i=1}^{n} |\overline{\mathcal{C}}_i|}{\sum_{i=1}^{n} |\mathcal{C}_i|}$ are as in section 2.4. Now, we can decompose the NBI as follows

$$\beta_i = (1 - \pi) \cdot \theta_i + \pi \cdot \varepsilon_i \tag{9}$$

which is the same as in equation (4).

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